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Characterization of tricyclic graphs with exactly two Q-main eigenvalues*

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Abstract: The signless Laplacian matrix of a graph G is defined to be the sum of its adjacency matrix and degree diagonal matrix, and its eigenvalues are called Q-eigenvalues of G. A Q-eigenvalue of a graph G is called a Q-main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. Chen and Huang [L. Chen, Q.X. Huang, Trees, unicyclic graphs and bicyclic graphs with exactly two Q-main eigenvalues, submitted for publication] characterized all trees, unicylic graphs and bicyclic graphs with exactly two main Q-eigenvalues, respectively. As a continuance of it, in this paper, all tricyclic graphs with exactly two Q-main eigenvalues are characterized.

Keywords: Signless Laplacian; Q-Main eigenvalue; Tricyclic graph

AMS subject classification: 05C50, 15A18

1. Introduction

Let $G = (V_G, E_G)$ be a simple connected graph with vertex set $V_G = \{v_1, \dots, v_n\}$ and edge set $E_G \neq \emptyset$. The adjacency matrix $A = A(G) = (a_{ij})$ of G is an $n \times n$ symmetric matrix with $a_{ij} = 1$ if and only if v_i, v_j are adjacent and 0 otherwise. Since G has no loops, the main diagonal of A contains only 0's. Suppose the valence or degree of vertex v_i equals $d_G(v_i)$ (or d_i) for $i = 1, \dots, n$, and let $D = D(G)$ be the diagonal matrix whose (i, i)-entry is $d_i, i = 1, 2, \dots, n$. The matrix $Q(G) = D(G) + A(G)$ has been called the *signless Laplacian matrix* of G. Recently, this matrix attracts more and more researchers' attention. For survey papers on this matrix the reader is referred to [1, 2, 3]. The eigenvalues and Q-eigenvalues of G are those of $A(G)$ and $Q(G)$, respectively. An eigenvalue (Q-eigenvalue) of a graph G is called a *main eigenvalue* (*Q-main eigenvalue*) if it has an eigenvector the sum of whose entries is not equal to zero. The Perron-Frobenius theorem implies that the largest eigenvalue and Q-eigenvalue of G are always main.

A vertex of a graph G is said to be *pendant* if its degree is one. Let $PV(G)$ be the set of all pendants of G. Throughout the text we denote by P_n and C_n the path and cycle on n vertices, respectively. Denote by $N(v)$ (or $N_G(v)$) the set of all neighbors of v in G. $c(G) = |E_G| - |V_G| + 1$ is said to be *cyclomatic number* of a connected graph G. In particular, G will be a tree, unicyclic graph, bicyclic graph, or tricyclic graph if $c(G) = 0, 1, 2$ or 3. Based on [15, 16, 17, 18, 19, 20, 21, 22] we know that a tricyclic graph G contains at least 3 cycles and at most 7 cycles, furthermore, there do not exist 5 cycles in G. Denote the set of tricyclic graphs on n vertices by $\mathscr{T}_n$. Then let $\mathscr{T}_n = \mathscr{T}_n^3 \bigcup \mathscr{T}_n^4 \bigcup \mathscr{T}_n^6 \bigcup \mathscr{T}_n^7$, where $\mathscr{T}_n^i$ denotes the set of tricyclic graphs in $\mathscr{T}_n$ with exact i cycles for $i = 3, 4, 6, 7$.

Let G be a connected graph, and $\widetilde{G}$ be the subgraph of G which is obtained from G by deleting its pendant vertex (if any) continuously until there is no any pendant vertex left. Obviously, $\widetilde{G}$ is a connected proper subgraph of G if G has non-empty pendant vertex set $PV(G)$ and $\widetilde{G} = G$ otherwise. We call $\widetilde{G}$ the *base* of G, and the vertex in $V_{\widetilde{G}}$ the *internal vertex* of G. If G contains a cycle with $PV(G) \neq \emptyset$, the longest path $P_G = v_0v_1 \dots v_k$ between $PV(G)$ and $V_{\widetilde{G}}$ (i.e., $v_0 \in PV(G), v_i \notin V_{\widetilde{G}} (i = 1, \dots, k-1)$ and $v_k \in V_{\widetilde{G}}$) is called the *longest pendant path*. $T_1, T_2, \dots, T_{15}$ are all the bases of tricyclic graphs depicted in Fig. 1. Throughout the context, we use $\widetilde{G} = T_i$ to mean that $\widetilde{G}$ has the same cycle arrangements and the same labelled vertices as that of T_i for $i = 1, 2, \dots, 15$.

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Call a path $P = u_0u_1 \dots u_k$ $(k \geq 1)$ an *internal path* of G if $d_G(u_0), d_G(u_k) \geq 3$ and $d_G(u_i) = 2$ for $1 \leq i \leq k-1$. Obviously, there are two types of internal paths: $u_0 \neq u_k (k \geq 1)$ and $u_0 = u_k (k \geq 3)$. For convenience, in what follows we call the former *internal path* and the latter *internal cycle*.

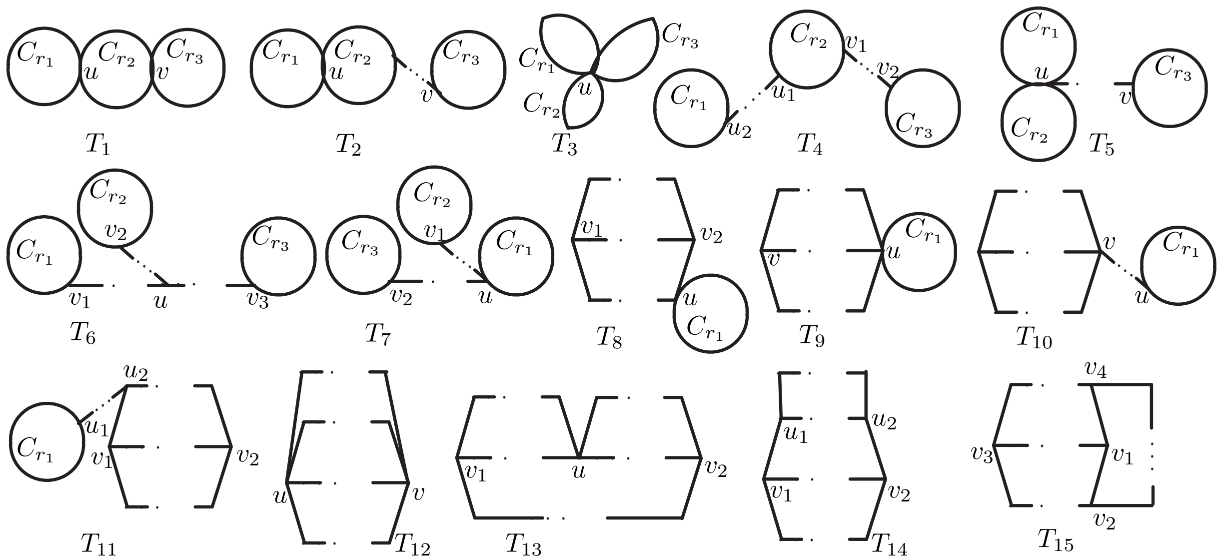


Figure 1: Graphs $T_1, T_2, \dots, T_{15}$: all the bases of tricyclic graphs.

There are some literatures on main eigenvalues of adjacency matrix $A(G)$, there is a survey in [4] related to main eigenvalues of a graph. A long standing problem posed by Cvetković (see [5]) is to characterize graphs with exactly k $(k \geq 2)$ main eigenvalues. The graphs with $c(G) \leq 3$ that have exactly two main eigenvalues are completely characterized (see, e.g. [6, 7, 8, 9, 10, 14]). Motivated by these works, Chen and Huang [13] obtained the following results on graphs with Q-main eigenvalues:

Theorem 1.1 ([13])**.** *G contains just one Q-main eigenvalue if and only if G is regular.*

Theorem 1.2 ([13])**.** *Let G be a graph with signless Laplacian Q. Then G has exactly two Q-mian eigenvalues if and only if there exist a unique pair of integers a and b such that for any $v \in V_G$*

$$\sum_{u \in N_G(v)} d_G(u) = a d_G(v) + b - d_G(v)^2 \tag{1.1}$$

For convenience, let $\mathscr{G}_{a,b}$ be the set of the connected graphs that satisfying (1.1), where (a, b) is the parameters of $G \in \mathscr{G}_{a,b}$. We know that $a > 0$ and $b \leq 0$ from [13].

Theorem 1.3 ([13])**.** *Let $G \in \mathscr{G}_{a,b}$ with pendant vertex, then $a + b - 1 \geq 2$. In particular, if $v \in V_G$ is a pendant vertex and u is the unique neighbor of v, then $d_G(u) = a + b - 1$.*

Theorem 1.4 ([13])**.** *Let $G \in \mathscr{G}_{a,b}$ with pendant vertex, and $P_{k+1} = v_0v_1v_2 \dots v_k$ be a longest pendant path in G. If G contains a cycle, then $1 \leq k \leq 2$. Moreover, if $k = 2$, then $b \leq -2$ and $b^2a^2 - (-2b^3 + 5b^2 + b + d_{\widetilde{G}}(v_2))a + b^4 - 5b^3 + 5b^2 + 2b - d_{\widetilde{G}}(v_2)b + 3d_{\widetilde{G}}(v_2) \leq 0$.*

In particular, Chen and Huang [13] characterized all trees, unicylic graphs and bicyclic graphs with exactly two main Q-eigenvalues, respectively. As a continuance of it, in this paper we consider Q-main eigenvalues of tricyclic graphs. We are to characterize all tricyclic graphs with exactly two Q-main eigenvalues in this paper. The organization of this work is as follows: In Section 2, all tricyclic graphs without pandants having exactly two

Q-main eigenvalues are determined. In Section 3, all tricyclic graphs with pendants having exactly two Q-main eigenvalues are identified.

Further on we will need the following lemmas.

Based on (1.1) the next lemma follows immediately.

Lemma 1.5. *If there exist $u, v \in V_G$ such that $d_G(u) = d_G(v)$, then $\sum_{w\in N_G(u)} d_G(w) = \sum_{w\in N_G(v)} d_G(w)$. Moreover, if $N_G(u) = \{u_1, u_2\}$ and $N_G(v) = \{v_1, v_2\}$, then $d_G(u_1) + d_G(u_2) = d_G(v_1) + d_G(v_2)$.*

Lemma 1.6. *Let $G \in \mathscr{G}_{a,b}$, $P = u_0u_1 \dots u_k$ be an internal path or an internal cycle in G. Then*

(i) *$k \leq 3$, and*

(ii) *If $k = 3$, then there exists no internal path of length 2, say $P_3 = v_0v_1v_2$, in G such that $d_G(v_0) = d_G(v_2) = d_G(u_0)$.*

(iii) *If $k = 3$, then $d_G(u_0) = d_G(u_3)$. Moreover, if there exists another internal path $P_4' = v_0v_1v_2v_3$ in G, then $d_G(v_0) = d_G(v_3) = d_G(u_0) = d_G(u_3)$.*

Proof. (i) On the contrary, suppose that $k \geq 4$. By definition, $d_G(u_0) = d_G(u_1) = d_G(u_2) = d_G(u_3) = 2$. Note that $d_G(u_1) = d_G(u_2) = 2$, hence by Lemma 1.5, we have

$$d_G(u_0) + 2 = \sum_{u\in N_G(u_1)} d_G(u) = \sum_{u\in N_G(u_2)} d_G(u) = 2 + 2 = 4,$$

which gives $d_G(u_0) = 2$, a contradiction to the condition that $d_G(u_0) \geq 3$.

(ii) If $k = 3$, on the contrary, suppose that there exists an internal path $P_3 = v_0v_1v_2$ in G such that $d_G(v_0) = d_G(v_2) = d_G(u_0)$. Note that $d_G(u_1) = d_G(v_1) = 2$, hence by Lemma 1.5, we have

$$d_G(u_0) + 2 = \sum_{u\in N_G(u_1)} d_G(u) = \sum_{u\in N_G(v_1)} d_G(u) = d_G(v_0) + d_G(v_2),$$

which gives $d_G(u_0) = 2$, a contradiction to the condition that $d_G(u_0) \geq 3$, as desired.

(iii) If P is an internal cycle, it's trivial since $u_0 = u_3$. If P is an internal path, note that $d_G(u_1) = d_G(u_2) = 2$, hence by Lemma 1.5, we have

$$d_G(u_0) + 2 = \sum_{u\in N_G(u_1)} d_G(u) = \sum_{u\in N_G(u_2)} d_G(u) = d_G(u_3) + 2,$$

which gives that $d_G(u_0) = d_G(u_3)$. Moreover, if there exists another internal path $P_4' = v_0v_1v_2v_3$ in G, note that $d_G(u_1) = d_G(v_1) = 2$, hence by Lemma 1.5, we have

$$d_G(u_0) + 2 = \sum_{u\in N_G(u_1)} d_G(u) = \sum_{u\in N_G(v_1)} d_G(u) = d_G(v_0) + 2,$$

which gives that $d_G(u_0) = d_G(v_0)$. Similarly, we can obtain that $d_G(u_3) = d_G(v_3)$. Then our result follows immediately. □

Lemma 1.7. *Let $G \in \mathscr{G}_{a,b}$ be a graph containing a cycle and let $P_{k+1} = v_0v_1 \dots v_k$ be a longest pendant path in G. Then we have $k = 1$.*

Proof. By Theorem 1.4 we have $1 \leq k \leq 2$. Clearly, $d_G(v_1) = a + b - 1$ by Theorem 1.3. In order to complete the proof, it suffices to show that $k \neq 2$. On the contrary, we assume that $k = 2$, then $v_2 \in V_{\widetilde{G}}$. From Theorem 1.4, we have $b \leq -2$ and

$$b^2a^2 - (-2b^3 + 5b^2 + b + r)a + (b^4 - 5b^3 + 5b^2 + 2b - rb + 3r) \leq 0, \tag{1.2}$$

where $r = d_{\widetilde{G}}(v_2) \geq 2$. The left sides of (1.2) may be viewed as quadratic equation of a and its discriminant is

$$\begin{aligned}\Delta &= (-2b^3 + 5b^2 + b + r)^2 - 4b^2(b^4 - 5b^3 + 5b^2 + 2b - rb + 3r) \\ &= b^4 + 2b^3 + b^2 - 2b^2 r + 2br + r^2.\end{aligned}$$

In order to complete the proof, it suffices to consider $\Delta \geq 0$. Note that $(b^2 - b - r)^2 - \Delta = -4b^3 > 0$, $(b^2 + b + r)^2 - \Delta = 4rb^2 > 0$, hence we have $b^2 - b - r > \sqrt{\Delta}$, $b^2 + b + r > \sqrt{\Delta}$.

By solving inequality (1.2), we have $a_1 \leq a \leq a_2$, where

$$a_1 = \frac{-2b^3 + 5b^2 + b + r - \sqrt{\Delta}}{2b^2}, \qquad a_2 = \frac{-2b^3 + 5b^2 + b + r + \sqrt{\Delta}}{2b^2}.$$

Notice that

$$\begin{aligned}a_1 &= \frac{-2b^3 + 5b^2 + b + r - \sqrt{\Delta}}{2b^2} > \frac{-2b^3 + 5b^2 + b + r - (b^2 + b + r)}{2b^2} = -b + 2, \\ a_2 &= \frac{-2b^3 + 5b^2 + b + r + \sqrt{\Delta}}{2b^2} < \frac{-2b^3 + 5b^2 + b + r + (b^2 - b - r)}{2b^2} = -b + 3,\end{aligned}$$

hence $-b + 2 < a < -b + 3$, we may get a contradiction since both of a and b are integers.

Hence, we obtain $k = 1$, as desired. $\square$

Given a graph $G \in \mathscr{G}_{a,b}$ with a cycle and pendants, if $u \in V_{\widetilde{G}}$ satisfies $d_G(u) \neq d_{\widetilde{G}}(u)$, then G must contain pendants attached to u by Lemma 1.7. Hence, $d_G(u) \in \{d_{\widetilde{G}}(u), a + b - 1\}$ for any $u \in V_{\widetilde{G}}$, and $d_G(u) = 1$ if $u \notin V_{\widetilde{G}}$. For $u \in V_{\widetilde{G}}$, u is called an *attached vertex* if it joins some pendant vertices and *non-attached vertex* otherwise. Let $u \in V_G$ be an attached vertex, then $d_G(u) = a + b - 1 > d_{\widetilde{G}}(u)$. Applying (1.1) at u, we have $\sum_{v \in N_{\widetilde{G}}(u)} d_G(v) + (d_G(u) - d_{\widetilde{G}}(u)) = a d_G(u) + b - d_G(u)^2$, which leads to $\sum_{v \in N_{\widetilde{G}}(u)} d_G(v) = -ab - b^2 + 2b + d_{\widetilde{G}}(u)$. If $b = 0$ then $\sum_{v \in N_{\widetilde{G}}(u)} d_G(v) = d_{\widetilde{G}}(u)$. On the other hand, since $d_G(v) \geq 2$ for $v \in N_{\widetilde{G}}(u)$, we have $\sum_{v \in N_{\widetilde{G}}(u)} d_G(v) \geq 2d_{\widetilde{G}}(u)$, a contradiction. Hence $b \leq -1$. Thus the next lemma follows immediately.

Lemma 1.8. *Let $G \in \mathscr{G}_{a,b}$ be a graph containing a cycle and pendants.*

(i) *If $u \in V_{\widetilde{G}}$, then $d_G(u) \in \{d_{\widetilde{G}}(u), a + b - 1\}$ and $d_G(u) = 1$ otherwise.*

(ii) *If $u \in V(\widetilde{G})$ is an attached vertex, then*

$$b \leq -1, \quad d_G(u) = a + b - 1 > d_{\widetilde{G}}(u), \qquad \sum_{v \in N_{\widetilde{G}}(u)} d_G(v) = -ab - b^2 + 2b + d_{\widetilde{G}}(u).$$

2. Tricyclic graphs without pendants having exactly two Q-main eigenvalues

In this section, we identify all the tricyclic graphs without pendants having exactly two Q-main eigenvalues.

Theorem 2.1. $G_1 \in \mathscr{G}_{8,-6}, G_2 \in \mathscr{G}_{7,-4}, G_3 \in \mathscr{G}_{9,-6}, G_4 \in \mathscr{G}_{7,-5}, G_5 \in \mathscr{G}_{6,-3}, G_6 \in \mathscr{G}_{6,-3}, G_7 \in \mathscr{G}_{8,-6}, G_8 \in \mathscr{G}_{7,-5}, G_9 \in \mathscr{G}_{6,-3}, G_{10} \in \mathscr{G}_{7,-2}, G_{11} \in \mathscr{G}_{8,-6}, G_{12} \in \mathscr{G}_{6,0}, G_{13} \in \mathscr{G}_{7,-4}, G_{14} \in \mathscr{G}_{7,-4}, G_{15} \in \mathscr{G}_{6,-2}, G_{16} \in \mathscr{G}_{6,-2}, G_{17} \in \mathscr{G}_{5,0}, G_{18} \in \mathscr{G}_{8,-7}, G_{19} \in \mathscr{G}_{7,-5}, G_{20} \in \mathscr{G}_{7,-5}, G_{21} \in \mathscr{G}_{6,-3}, G_{22} \in \mathscr{G}_{7,-4}, G_{23} \in \mathscr{G}_{8,-7}, G_{24} \in \mathscr{G}_{6,-2}, G_{25} \in \mathscr{G}_{7,-5}, G_{26} \in \mathscr{G}_{5,0}, G_{27} \in \mathscr{G}_{6,-3}$ *(see Fig. 2) are all the tricyclic graphs with no pendants having exactly two Q-main eigenvalues.*

Proof. Let $G \in \mathscr{G}_{a,b}$ be a tricyclic graph with no pendants having exactly two Q-main eigenvalues. Hence, $G \cong \widetilde{G} \in \mathscr{T}_n = \mathscr{T}_n^3 \bigcup \mathscr{T}_n^4 \bigcup \mathscr{T}_n^6 \bigcup \mathscr{T}_n^7$. We consider the following possible cases.

Case 1. $G \in \mathscr{T}_n^3$. In this case, G contains three cycles, say C_{r_1}, C_{r_2} and C_{r_3}. By Lemma 1.6(i), if C_{r_i} is an internal cycle, then $r_i = 3$.

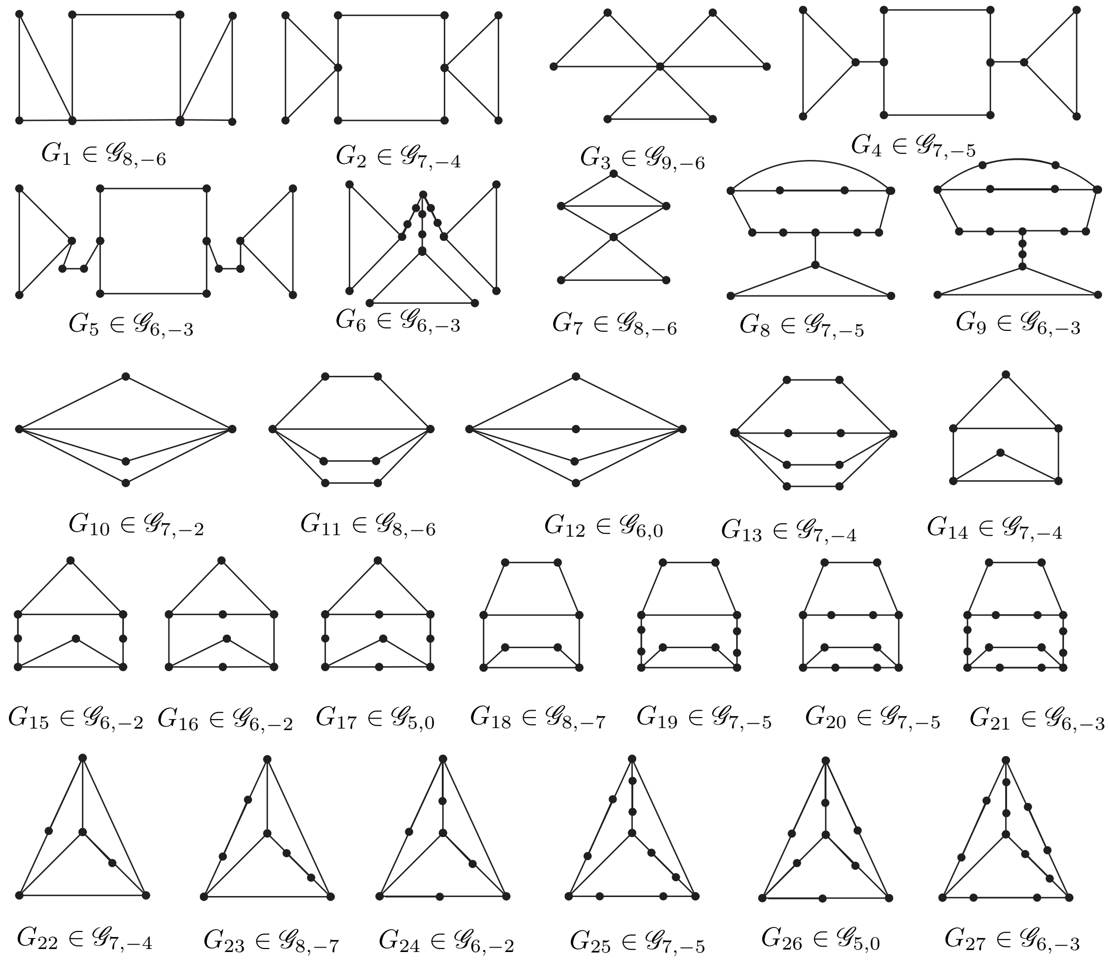


Figure 2: Graphs $G_1, G_2, \ldots, G_{27}$.

• $G = T_1$; see Fig. 1. We have $r_1 = r_3 = 3$. C_{r_2} consists of the two internal paths: P_{k_1+1} and P_{k_2+1} connect u, v, where $d_G(u) = d_G(v) = 4$. By Lemma 1.6(i), we have $k_1 \leq 3, k_2 \leq 3$. By Lemma 1.6(ii), we have $k_1 \neq 2, k_2 \neq 2$. Then without loss of generality, we assume $k_1 = 1, k_2 = 3$ or $k_1 = k_2 = 3$. It's simple to verify that $G \cong G_1 \in \mathscr{G}_{8,-6}$ if $k_1 = 1, k_2 = 3$, $G \cong G_2 \in \mathscr{G}_{7,-4}$ if $k_1 = k_2 = 3$, where G_1, G_2 are depicted in Fig. 2.

• $G = T_2$; see Fig. 1. We have $r_1 = r_3 = 3$. Suppose $C_{r_1} = us_1s_2u$, and $C_{r_3} = vt_1t_2v$. Note that $d_G(s_1) = d_G(t_1) = 2$, hence by Lemma 1.5, we have

$$2 + 4 = d_G(s_2) + d_G(u) = d_G(t_2) + d_G(v) = 2 + 3,$$

a contradiction.

• $G = T_3$; see Fig. 1. We have $r_1 = r_2 = r_3 = 3$. It's routine to check that $G \cong G_3 \in \mathscr{G}_{9,-6}$; see Fig. 2.

• $G = T_4$; see Fig. 1. We have $r_1 = r_3 = 3$. C_{r_2} consists of the two internal paths: P_{k_1+1} and P_{k_2+1} connect u_1 and v_1; one internal path P_{k_3+1} connects u_1, u_2 and one internal path P_{k_4+1} connects v_1 and v_2, where $d_G(u_1) = d_G(u_2) = d_G(v_1) = d_G(v_2) = 3$. By Lemma 1.6(i), we have $k_i \leq 3$, $i = 1, 2, 3, 4$. By Lemma 1.6(ii), we have $k_i \neq 2$, $i = 1, 2, 3, 4$. Then $k_3 = 1, 3, k_4 = 1, 3$, furthermore, we assume, without loss of generality, that $k_1 = 1, k_2 = 3$ or $k_1 = k_2 = 3$. If $k_1 = 1, k_2 = 3$, it's easy to prove that $k_3 \neq 1,\, 3, k_4 \neq 1,\, 3$. In fact, if $k_3 = 1$, based on Lemma 1.5, we have

$$2 + 2 + 3 = \sum_{u \in N_G(u_2)} d_G(u) = \sum_{u \in N_G(u_1)} d_G(w) = 2 + 3 + 3,$$

a contradiction; if $k_3 = 3$, based on Lemma 1.5, we have

$$2 + 2 + 2 = \sum_{u \in N_G(u_2)} d_G(u) = \sum_{u \in N_G(u_1)} d_G(w) = 2 + 2 + 3,$$

a contradiction. Hence, $k_3 \neq 1, 3$. Similarly, $k_4 \neq 1, 3$. If $k_1 = k_2 = 3$, it's impossible if $k_3 = 1, k_4 = 3$ or $k_3 = 3, k_4 = 1$. In fact, if $k_3 = 1, k_4 = 3$, based on Lemma 1.5, we have

$$2 + 2 + 3 = \sum_{u \in N_G(u_1)} d_G(u) = \sum_{u \in N_G(v_1)} d_G(w) = 2 + 2 + 2,$$

a contradiction. Similarly, $k_3 = 3, k_4 = 1$ gives a contradiction. It is routine to check that $G \cong G_4 \in \mathscr{G}_{7,-5}$ if $k_3 = k_4 = 1$; whereas $G \cong G_5 \in \mathscr{G}_{6,-3}$ if $k_3 = k_4 = 3$.

• $G = T_5$; see Fig. 1. We have $r_1 = r_2 = r_3 = 3$. Suppose $C_{r_1} = us_1s_2u$, and $C_{r_3} = vt_1t_2v$. Note that $d_G(s_1) = d_G(t_1) = 2$, based on Lemma 1.5, then we have

$$2 + 5 = d_G(s_2) + d_G(u) = d_G(t_2) + d_G(v) = 2 + 3,$$

which gives a contradiction.

• $G = T_6$; see Fig. 1. We have $r_1 = r_2 = r_3 = 3$. There exist three internal paths $P_{k_i+1}(i = 1, 2, 3)$ connecting u and v_i, respectively. By Lemma 1.6(i), $k_1, k_2, k_3 \leq 3$. Since $d_G(u) = d_G(v_1) = d_G(v_2) = d_G(v_3) = 3$, by Lemma 1.6(ii), $k_1, k_2, k_3 \neq 2$. Next we show that $k_1, k_2, k_3 \neq 1$. In fact, suppose to the contrary, assume $k_1 = 1$. Based on Lemma 1.5, we have

$$2 + 2 + 3 = \sum_{w \in N_G(v_1)} d_G(w) = \sum_{w \in N_G(u)} d_G(w) = 3 + \sum_{w \in N_G(u) \setminus \{v_1\}} d_G(w),$$

which is equivalent to $\sum_{w \in N_G(u) \setminus \{v_1\}} d_G(w) = 4$. It implies that $k_2 = k_3 = 3$. Based on Lemma 1.5, we have

$$2 + 2 + 3 = \sum_{w \in N_G(u)} d_G(w) = \sum_{w \in N_G(v_3)} d_G(w) = 2 + 2 + 2,$$

a contradiction. Hence, $k_1 \neq 1$. Similarly, $k_2, k_3 \neq 1$. Then $k_1 = k_2 = k_3 = 3$. It's simple to verify that $G \cong G_6 \in \mathscr{G}_{6,-3}$; see Fig. 2.

• $G = T_7$; see Fig. 1. We have $r_1 = r_2 = r_3 = 3$. Suppose $C_{r_1} = us_1s_2u$ and $C_{r_2} = v_1t_1t_2v_1$. Note that $d_G(s_1) = d_G(t_1) = 2$, based on Lemma 1.5, then we have

$$2 + 4 = d_G(s_2) + d_G(u) = d_G(t_2) + d_G(v) = 2 + 3,$$

which gives a contradiction.

Case 2. $G \in \mathscr{T}_n^4$. In this case, it is easy to see that G contains an internal cycle C_{r_1} with $r_1 = 3$.

• $G = T_8$; see Fig. 1. Suppose $C_{r_1} = uz_1z_2u$. There exist two internal paths P_{k_1+1} and P_{k_2+1} connecting v_1 and v_2, one internal path P_{k_3+1} connecting u and v_1, and one internal path P_{k_4+1} connecting u and v_2. By Lemma 1.6(i), we have $k_i \leq 3$ for $i = 1, 2, 3, 4$. First we show that $k_3 = k_4 = 1$. By Lemma 1.6(iii), we have that $k_3 \neq 3$. If $k_3 = 2$, note that $d_G(z_1) = d_G(x_1) = 2$, then based on Lemma 1.5, we have

$$2 + 4 = d_G(z_2) + d_G(u) = d_G(v_1) + d_G(u) = 3 + 4,$$

a contradiction. Thus, $k_3 = 1$; similarly, $k_4 = 1$. Note that $d_G(u) = 4, d_G(v_1) = d_G(v_2) = 3$, hence by Lemma 1.6(iii), we have that $k_1, k_2 \neq 3$. Without loss of generality, we assume $k_1 = 1, k_2 = 2$ or $k_1 = k_2 = 2$. It's simple

to verify that $G \cong G_7 \in \mathscr{G}_{8,-6}$ if $k_1 = 1, k_2 = 2$. While if $k_1 = k_2 = 2$, applying (1.1) gives no integer solution, a contradiction.

• $G = T_9$; see Fig. 1. There exist three internal paths $P_{k_1+1}, P_{k_2+1}, P_{k_3+1}$ connecting u and v. By Lemma 1.6(i), $k_1, k_2, k_3 \leq 3$. By Lemma 1.6(iii), $k_1, k_2, k_3 \neq 3$. Without loss of generality we assume that $k_1 = 2$ and $P_{k_1+1} = us_1v$. Let $C_{r_1} = ut_1t_2u$. Note that $d_G(t_1) = d_G(s_1) = 2$, based on Lemma 1.5, we have

$$2 + 5 = d_G(t_2) + d_G(u) = d_G(v) + d_G(u) = 3 + 5,$$

a contradiction.

• $G = T_{10}$; see Fig. 1. By a similar discussion as in the proof of $G = T_9$, we may obtain that there does not exist such graph in $\mathscr{G}_{a,b}$, we omit the procedure here.

• $G = T_{11}$; see Fig. 1. In this subcase, there exist two internal paths P_{k_1+1} and P_{k_2+1} connecting v_1, v_2; one internal path P_{k_3+1} connecting v_1, u_2, one internal path P_{k_4+1} connecting v_2, u_2, and one internal path P_{k_5+1} connecting u_1, u_2. Note that $d_G(u_1) = d_G(u_2) = d_G(v_1) = d_G(v_2) = 3$. By Lemma 1.6(i), $k_i \leq 3\,(i = 1, 2, 3, 4, 5)$. By Lemma 1.6(ii), $k_i \neq 2\,(i = 1, 2, 3, 4, 5)$. If $k_5 = 1$, then based on Lemma 1.5, we have $k_3 = k_4 = 3, k_1 = 1, k_2 = 3$ and it's simple to verify that $G \cong G_8 \in \mathscr{G}_{7,-5}$; see Fig. 2. If $k_5 = 3$, similarly as above we obtain that $k_1 = k_2 = k_3 = k_4 = 3$ and it's simple to verify that $G \cong G_9 \in \mathscr{G}_{6,-3}$; see Fig. 2.

Case 3. $G \in \mathscr{T}_n^6$.

• $G = T_{12}$; see Fig. 1. In this subcase, G consists of four internal paths: $P_{k_1+1}, P_{k_2+1}, P_{k_3+1}, P_{k_4+1}$ connecting u, v. By Lemma 1.6(i), $k_i \leq 3$ for $i = 1, 2, 3, 4$. Note that $d_G(u) = d_G(v) = 4$. By Lemma 1.6(ii), if there exist $i_0 \in \{1, 2, 3, 4\}$ such that $k_{i_0} = 3$, then $k_i \neq 2$ for each $i \in \{1, 2, 3, 4\} \setminus \{i_0\}$. Without loss of generality, we may assume $k_1 = 1, k_2 = k_3 = k_4 = 2$, or $k_1 = 1, k_2 = k_3 = k_4 = 3$, or $k_1 = k_2 = k_3 = k_4 = 2$, or $k_1 = k_2 = k_3 = k_4 = 3$. It's simple to verify that $G \cong G_{10} \in \mathscr{G}_{7,-2}$ if $k_1 = 1, k_2 = k_3 = k_4 = 2$; $G \cong G_{11} \in \mathscr{G}_{8,-6}$ if $k_1 = 1, k_2 = k_3 = k_4 = 3$; $G \cong G_{12} \in \mathscr{G}_{6,0}$ if $k_1 = k_2 = k_3 = k_4 = 2$; $G \cong G_{13} \in \mathscr{G}_{7,-4}$ if $k_1 = k_2 = k_3 = k_4 = 3$, where G_{10}, G_{11}, G_{12} and G_{13} are depicted in Fig. 2.

• $G = T_{13}$; see Fig. 1. By a similar discussion as in the proof of $G = T_9$, we may obtain that there does not exist such graph in $\mathscr{G}_{a,b}$, we omit the procedure here.

• $G = T_{14}$; see Fig. 1. In this subcase, G consists of six internal paths: two paths P_{k_1+1}, P_{k_2+1} connect u_1, u_2, two paths P_{k_3+1}, P_{k_4+1} connect v_1, v_2, one path P_{k_5+1} connects u_1, v_1 and one path P_{k_6+1} connects u_2, v_2. By Lemma 1.6(i), $k_i \leq 3\,(i = 1, 2, 3, 4, 5, 6)$. Note that $d_G(u_1) = d_G(u_2) = d_G(v_1) = d_G(v_2) = 3$, by Lemma 1.6(ii), if there exists $i_0 \in \{1, 2, 3, 4, 5, 6\}$ such that $k_{i_0} = 3$, then $k_i \neq 2$ for each $i \in \{1, 2, 3, 4, 5, 6\} \setminus \{i_0\}$.

If $k_1 = 1, k_2 = 2$, then $k_3 = 1, k_4 = 2$. Furthermore, if $k_5 = 1$, then $k_6 = 1$, and it's simple to verify that $G \cong G_{14} \in \mathscr{G}_{7,-4}$; If $k_5 = 2$, then $k_6 = 2$, and it's simple to verify that $G \cong G_{15} \in \mathscr{G}_{6,-2}$, where G_{14} and G_{15} are depicted in Fig. 2.

If $k_1 = k_2 = 2$, then $k_3 = k_4 = 2$. Furthermore, if $k_5 = 1$, then $k_6 = 1$, and it's simple to verify that $G \cong G_{16} \in \mathscr{G}_{6,-2}$; If $k_5 = 2$, then $k_6 = 2$, and it's simple to verify that $G \cong G_{17} \in \mathscr{G}_{5,0}$, where G_{16} and G_{17} are depicted in Fig. 2.

If $k_1 = 1, k_2 = 3$, then $k_3 = 1, k_4 = 3$. Furthermore, if $k_5 = 1$, then $k_6 = 1$, and it's simple to verify that $G \cong G_{18} \in \mathscr{G}_{8,-7}$; If $k_5 = 2$, then $k_6 = 2$, and it's simple to verify that $G \cong G_{19} \in \mathscr{G}_{7,-5}$, where G_{18} and G_{19} are depicted in Fig. 2.

If $k_1 = k_2 = 3$, then $k_3 = k_4 = 3$. Furthermore, if $k_5 = 1$, then $k_6 = 1$, and it's simple to verify that $G \cong G_{20} \in \mathscr{G}_{7,-5}$; If $k_5 = 3$, then $k_6 = 3$, and it's simple to verify that $G \cong G_{21} \in \mathscr{G}_{6,-3}$, where G_{20} and G_{21} are depicted in Fig. 2.

Case 4. $G \in \mathscr{T}_n^7$.

In this case, $G = T_{15}$ (see Fig. 1), hence G consists of six internal paths: one path $P_{k_{i-1}+1}$ connects $v_1, v_i\,(i = 2, 3, 4)$, one path P_{k_4+1} connects v_2, v_3, one path P_{k_5+1} connects v_3, v_4, and one path P_{k_6+1} connects v_2, v_4. By Lemma 1.6(i), $k_i \le 3\,(i = 1, 2, 3, 4, 5, 6)$. Note that $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_4) = 3$. By Lemma 1.6(ii), if there exists $i_0 \in \{1, 2, 3, 4, 5, 6\}$ such that $k_{i_0} = 3$, then $k_i \ne 2$ for each $i \in \{1, 2, 3, 4, 5, 6\}\backslash\{i_0\}$.

If $k_1 = k_2 = k_3 = 1$. It's simple to verify that $k_4 = k_5 = k_6 = 1$, which implies that G is a regular graph. By Theorem 1.1, G contains just one Q-main eigenvalue, a contradiction.

If $k_1 = k_2 = 1$. Furthermore, if $k_3 = 2$, it's simple to verify that $k_4 = 2, k_5 = k_6 = 1$ and $G \cong G_{22} \in \mathscr{G}_{7,-4}$, if $k_3 = 3$, it's simple to verify that $k_4 = 3, k_5 = k_6 = 1$ and $G \cong G_{23} \in \mathscr{G}_{8,-7}$. Here G_{22}, G_{23} are depicted in Fig. 2.

If $k_1 = 1$. Furthermore, if $k_2 = k_3 = 2$, it's simple to verify that $k_4 = k_6 = 2, k_5 = 1$ and $G \cong G_{24} \in \mathscr{G}_{6,-2}$, if $k_2 = k_3 = 3$, it's simple to verify that $k_4 = k_6 = 3, k_5 = 1$ and $G \cong G_{25} \in \mathscr{G}_{7,-5}$. Here G_{24}, G_{25} are depicted in Fig. 2.

If $k_1 = k_2 = k_3 = 2$, it's simple to verify that $k_4 = k_5 = k_6 = 2$ and $G \cong G_{26} \in \mathscr{G}_{5,0}$. If $k_1 = k_2 = k_3 = 3$, it's simple to verify that $k_4 = k_5 = k_6 = 3$ and $G \cong G_{27} \in \mathscr{G}_{6,-3}$. Here, G_{26}, G_{27} are depicted in Fig. 2.

This completes the proof. $\square$

3. Tricyclic graphs with pendants having exactly two Q-main eigenvalues

In this section, we identify all the tricyclic graphs with pendants having exactly two Q-main eigenvalues.

Lemma 3.1. *Given a tricyclic graph $G \in \mathscr{G}_{a,b}$ with pendants. If $C_p = vu_1u_2\ldots u_{p-1}v$ is an internal cycle of $\widetilde{G}$ with $N_{\widetilde{G}}(v) = \{w, u_1, u_{p-1}\}$, then $d_G(v) = 3$, $p = 3$ and $G \in \mathscr{G}_{6,-1}$. Moreover, $d_G(u_1) = d_G(w) = 2$, $d_G(u_{p-1}) = a + b - 1 = 4$.*

Proof. For convenience, let $r = d_G(w)$. Since G is in $\mathscr{T}_n$, we have $r \in \{2, 3, 4, 5, a + b - 1\}$. Note that $d_G(u_1), d_G(u_{p-1}) \in \{2, a + b - 1\}$, let

$$t = |\{x : d_G(x) = 2,\, x \in \{u_1, u_{p-1}\}\}|, \tag{3.1}$$

hence, $t = 0, 1, 2$. We consider the following two possible cases according to $d_G(v)$.

Case 1. $d_G(v) = a + b - 1 > 3$. In this case, by Lemma 1.8 $b \le -1$, hence $a \ge 6$.

If $d_G(u_1) = 2$, applying (1.1) at u_1, we have $d_G(u_2) = a - 3 \in \{2, a + b - 1\}$. Notices that $a \ge 6$, hence $a - 3 \ne 2$, i.e. $d_G(u_2) = a - 3 = a + b - 1$, from which we get that $b = -2$; if $d_G(u_1) = a + b - 1$, applying (1.1) at u_1, we have $d_G(u_2) = -ab - b^2 - a + b + 3$.

Applying Lemma 1.8(ii) at v yields

$$d_G(u_1) + d_G(u_{p-1}) + d_G(w) = -ab - b^2 + 2b + 3. \tag{3.2}$$

In view of (3.1) we have

$$2t + (a + b - 1)(2 - t) + r = -ab - b^2 + 2b + 3, \quad (t = 0, 1, 2) \tag{3.3}$$

- $t = 0$. In this subcase, (3.3) is equivalent to $ab + b^2 + 2a = 5 - r$. As $d_G(u_1) = a + b - 1$, we have

$$ab + b^2 + 2a = 5 - r \quad \text{and} \quad d_G(u_2) = -ab - b^2 - a + b + 3.$$

with

$$r \in \{2, 3, 4, 5, a + b - 1\}, \quad a + b > 4 \tag{3.4}$$

Note that $d_G(u_2) \in \{2, a+b-1\}$. If $d_G(u_2) = 2$, we get $a+b = 4-r$, a contradiction to (3.4); if $d_G(u_2) = a+b-1$, we get $r = 1$, a contradiction to (3.4) either.

• $t = 1$. In this subcase, (3.3) gives $ab + b^2 + a - b = 2 - r$. Without loss of generality, we assume that $d_G(u_1) = 2$. Then we have $b = -2$ with $d_G(u_2) = a + b - 1$. Hence, $a = 4 + r$. Note that $a + b - 1 > 3$, then we get $r \in \{3, 4, 5\}$. If $r = 3$, then $(a, b) = (7, -2)$ and $d_G(u_2) = 4$. Applying (1.1) at u_2, it is easy to get that $d_G(u_3) = 6 \notin \{2, a + b - 1 = 4\}$, a contradiction. If $r = 4$, then $(a, b) = (8, -2)$ and $d_G(u_2) = 5$. We apply (1.1) at u_2 to get that $d_G(u_3) = 8 \notin \{2, a + b - 1 = 5\}$, a contradiction. If $r = 5$, then $(a, b) = (9, -2)$ and $d_G(u_2) = 6$. We apply (1.1) at u_2 to get that $d_G(u_3) = 10 \notin \{2, a + b - 1 = 6\}$, a contradiction.

• $t = 2$. In this subcase, (3.3) gives $ab + b^2 - 2b + 1 + r = 0$. As $d_G(u_1) = 2$, we have $b = -2$ with $d_G(u_2) = a + b - 1$. Hence, $2a = 9 + r$. Note that a is an integer and $a + b - 1 > 3$, then we get $r = 5$. Therefore, $(a, b) = (7, -2)$ and $d_G(u_2) = 4$. We apply (1.1) at u_2 to get that $d_G(u_3) = 6 \notin \{2, a+b-1 = 4\}$, a contradiction.

Case 2. $d_G(v) = 3$. In this case, $a + b - 1 \geq 3$.

If $d_G(u_1) = 2$, applying (1.1) at u_1, we get that $d_G(u_2) = 2a + b - 7$; if $d_G(u_1) = a + b - 1$, applying (1.1) at u_1, we get that $d_G(u_2) = -ab - b^2 + 2b - 1$.

Applying (1.1) at v yields

$$d_G(u_1) + d_G(u_{p-1}) + d_G(w) = 3a + b - 9. \tag{3.5}$$

In view of (3.1) we have

$$2t + (a + b - 1)(2 - t) + r = 3a + b - 9, \qquad (t = 0, 1, 2) \tag{3.6}$$

• $t = 0$. By (3.6), $a - b = 7 + r$. Since $d_G(u_1) = a + b - 1$, we have

$$a - b = 7 + r \quad \text{and} \quad d_G(u_2) = -ab - b^2 + 2b - 1$$

with $d_G(u_2) \in \{2, a + b - 1\}$. For $r \in \{2, a + b - 1\}$, the equation system gives that $(a, b) = (6, -3)$. However, $a + b - 1 = 2 < 3$, a contradiction. Moreover, it's easy to verify that the equation system above has no integer solutions for $r \in \{3, 4, 5\}$, a contradiction.

• $t = 1$. By (3.6) $2a = 10 + r$. Note that a is an integer, then we have $r \in \{2, 4, a + b - 1\}$. Without loss of generality, we assume that $d_G(u_1) = 2$, then

$$2a = 10 + r \quad \text{and} \quad d_G(u_2) = 2a + b - 7$$

with $d_G(u_2) \in \{2, a + b - 1\}$. Since a and b are both integers satisfying $a + b - 1 \geq 3$, it's routine to check that only $d_G(u_2) = a + b - 1$ with $r = 2$ holds, which gives that $(a, b) = (6, -1)$. Then $d_G(u_2) = 4$. Applying (1.1) at u_2, we have $d_G(u_3) = 3$. It implies that $u_3 = v$ and $p = 3$. Note that $d_G(w) = r = 2$.

• $t = 2$. By (3.6) $3a + b = 13 + r$. Since $d_G(u_1) = 2$, we have

$$3a + b = 13 + r \quad \text{and} \quad d_G(u_2) = 2a + b - 7$$

with $d_G(u_2) \in \{2, a + b - 1\}$.

For $r \in \{2, a+b-1\}$, it's routine to check that only $d_G(u_2) = a+b-1$ with $r = a+b-1$ holds, which gives that $a = 6$ and $b \in \{-1, -2\}$ since a and b are both integers satisfying $a+b-1 \geq 3$. First we consider $(a, b) = (6, -1)$. Then we have $d_G(u_2) = 4$. Applying (1.1) at u_2 yields $d_G(u_3) = 3 \notin \{2, a + b - 1 = 4\}$, a contradiction. Now we consider $(a, b) = (6, -2)$. Then we have $d_G(u_2) = 3$. Applying (1.1) at u_2 yields $d_G(u_3) = 4 \notin \{2, a + b - 1 = 3\}$, a contradiction.

For $r = 3$, if $d_G(u_2) = 2$, then $(a, b) = (7, -5), a+b-1 = 1$, a contradiction. So we have $d_G(u_2) = a+b-1$. Thus $(a, b) = (6, -2)$ and $d_G(u_2) = 3$. Applying (1.1) at u_2, we have $d_G(u_3) = 4 \notin \{2, a + b - 1 = 3\}$, a contradiction.

For $r = 4$, if $d_G(u_2) = 2$, then $(a, b) = (8, -7), a + b - 1 = 0$, a contradiction. Hence, we have $d_G(u_2) = a + b - 1$, thus $(a, b) = (6, -1)$ and $d_G(u_2) = 4$. Applying (1.1) at u_2, we have $d_G(u_3) = 3 \notin \{2, a + b - 1 = 4\}$. For $r = 5$, if $d_G(u_2) = 2$, then $(a, b) = (9, -9)$, a contradiction. Hence, we have $d_G(u_2) = a + b - 1$, thus $(a, b) = (6, 0)$, a contradiction to $b \leq -1$.

This completes the proof. □

Lemma 3.2. *Let $G \in \mathscr{G}_{a,b}$ be a tricyclic graph with pendants. If $C_p = vu_1 \dots u_{p-1}v$ is an internal cycle of $\widetilde{G}$ with $N_{\widetilde{G}}(v) = \{u_1, u_{p-1}, w_1, w_2\}$, then $d_G(v) = 4$, $p = 3$ and $G \in \mathscr{G}_{7,-1}$. Moreover, $d_G(u_1) = d_G(w_1) = d_G(w_2) = 2$, $d_G(u_{p-1}) = a + b - 1 = 5$.*

Proof. For convenience, let $d_1 = d_G(w_1), d_2 = d_G(w_2)$. Note that $G \in \mathscr{T}_n$, we have $d_1, d_2 \in \{2, 3, 4, a + b - 1\}$. We consider the following two possible cases according to $d_G(v)$.

Case 1. $d_G(v) = a + b - 1 > 4$. Note that $b \leq -1$, hence in this case, $a \geq 7$.

If $d_G(u_1) = 2$, we apply (1.1) at u_1 to get $d_G(u_2) = a - 3 \in \{2, a + b - 1\}$. As $a \geq 7$, we obtain $a - 3 \neq 2$. Thus, $d_G(u_2) = a - 3 = a + b - 1$, i.e., $b = -2$; If $d_G(u_1) = a + b - 1$, applying (1.1) at u_1, we get $d_G(u_2) = -ab - b^2 - a + b + 3$.

Applying Lemma 1.8(ii) at v yields

$$d_G(u_1) + d_G(u_{p-1}) + d_1 + d_2 = -ab - b^2 + 2b + 4. \tag{3.7}$$

We first consider $d_1 \in \{2, a + b - 1\}$. Let

$$t_1 = |\{x : d_G(x) = 2, \, x \in \{u_1, u_{p-1}, w_1\}\}|,$$

hence $t_1 = 0, 1, 2, 3$. Together with (3.7) we have

$$2t_1 + (a + b - 1)(3 - t_1) + d_2 = -ab - b^2 + 2b + 4, \qquad (t_1 = 0, 1, 2, 3). \tag{3.8}$$

- $t_1 = 0$. By (3.8), $ab + b^2 + 3a + b = 7 - d_2$. As $d_G(u_1) = a + b - 1$, we have

$$ab + b^2 + 3a + b = 7 - d_2 \quad \text{and} \quad d_G(u_2) = -ab - b^2 - a + b + 3.$$

with $d_G(u_2) \in \{2, a + b - 1\}$. It's routine to check that there is no integer solution such that $a + b - 1 > 4$ since $d_2 \in \{2, 3, 4, a + b - 1\}$, a contradiction.

- $t_1 = 1$. By (3.8), $ab + b^2 + 2a = 4 - d_2$. If $d_G(u_1) = 2$, then $b = -2$, which gives $d_2 = 0$, a contradiction; If $d_G(u_1) = a + b - 1$, then we have

$$ab + b^2 + 2a = 4 - d_2 \quad \text{and} \quad d_G(u_2) = -ab - b^2 - a + b + 3.$$

with $d_G(u_2) \in \{2, a + b - 1\}$. It's routine to check that there is no integer solution such that $a + b - 1 > 4$ since $d_2 \in \{2, 3, 4, a + b - 1\}$, a contradiction.

- $t_1 = 2$. By (3.8), $ab + b^2 + a - b = 1 - d_2$. Without loss of generality, we assume that $d_G(u_1) = 2$, then we have $b = -2$ with $d_G(u_2) = a + b - 1$. So $a = 5 + d_2$. It's obvious that $d_2 \neq a + b - 1$. Moreover, since $a + b - 1 > 4$, we have $a \geq 8$, which implies $d_2 \geq 3$. Then $d_2 \in \{3, 4\}$. If $d_2 = 3$, then $(a, b) = (8, -2)$ and $d_G(u_2) = 5$. We apply (1.1) at u_2 to get $d_G(u_3) = 8 \notin \{2, a + b - 1 = 5\}$, a contradiction. If $d_2 = 4$, then $(a, b) = (9, -2)$ and $d_G(u_2) = 6$. We apply (1.1) at u_2 to get $d_G(u_3) = 10 \notin \{2, a + b - 1 = 6\}$, a contradiction.
- $t_1 = 3$. By (3.8), $ab + b^2 - 2b + d_2 + 2 = 0$. Since $d_G(u_1) = 2$, we have $b = -2$. So $2a = 10 + d_2$. Since $a + b - 1 > 4$ and a is an integer, it's easy to verify that $d_2 \notin \{2, 3, 4, a + b - 1\}$, a contradiction.

Hence, we conclude that $d_1 \notin \{2, a + b - 1\}$. Similarly, $d_2 \notin \{2, a + b - 1\}$. Hence, $d_1, d_2 \in \{3, 4\}$. Moreover, by Fig. 1, we know that $d_1 = d_2 = 3$ and $\widetilde{G} \cong T_7, T_8$. Let

$$t_2 = |\{x : d_G(x) = 2, x \in \{u_1, u_{p-1}\}\}|,$$

hence, $t_2 = 0, 1, 2$. Together with (3.7) we have

$$2t_2 + (a+b-1)(2-t_2) + 3 + 3 = -ab - b^2 + 2b + 4, \qquad (t_2 = 0, 1, 2) \tag{3.9}$$

If $t_2 = 0$, then (3.9) gives $ab + b^2 + 2a = 0$. Since $d_G(u_1) = a+b-1$, we have $d_G(u_2) = -ab - b^2 - a + b + 3 \in \{2, a+b-1\}$, which gives no integer solution such that $a+b-1 > 4$, a contradiction.

If $t_2 = 1$, then (3.9) gives $ab + b^2 + a - b + 3 = 0$. Without loss of generality, we assume that $d_G(u_1) = 2$, then we have $b = -2$ with $d_G(u_2) = a+b-1$, which gives that $(a, b) = (9, -2)$ and $d_G(u_2) = 6$. We apply (1.1) at u_2 to get $d_G(u_3) = 10 \notin \{2, a+b-1 = 6\}$, a contradiction.

If $t_2 = 2$, then (3.9) gives $ab + b^2 - 2b + 6 = 0$. Since $d_G(u_1) = 2$, we have $b = -2$, which gives that $(a, b) = (7, -2)$, a contradiction to the assumption $a+b-1 > 4$.

Case 2. $d_G(v) = 4$. In this case, $a+b-1 \geq 3$.

If $d_G(u_1) = 2$, we apply (1.1) at u_1 to get $d_G(u_2) = 2a + b - 8$; if $d_G(u_1) = a+b-1$, we apply (1.1) at u_1 to get $d_G(u_2) = -ab - b^2 + 2b - 2$.

Applying (1.1) at v, we have

$$d_G(u_1) + d_G(u_{p-1}) + d_G(w_1) + d_G(w_2) = 4a + b - 16. \tag{3.10}$$

We first consider $d_1, d_2 \in \{2, a+b-1\}$. Let

$$t_1' = |\{x : d_G(x) = 2,\ x \in \{u_1, u_{p-1}, w_1, w_2\}\}|,$$

hence $t_1' = 0, 1, 2, 3, 4$. Together with (3.10) we have

$$2t_1' + (a+b-1)(4-t_1') = 4a + b - 16, \qquad (t_1' = 0, 1, 2, 3, 4)$$

If $t_1' = 0, 1$, let $d_G(u_1) = a+b-1$; if $t_1' = 2$, then $d_G(u_1) \in \{2, a+b-1\}$; if $t_1' = 3, 4$, let $d_G(u_1) = 2$. It's routine to check that only $d_G(u_2) = a+b-1$ with $t_1' = 3$ holds, which gives that $a = 7$ and $b \in \{-1, -2, -3\}$ since a and b are both integers satisfying $a+b-1 \geq 3$.

For $(a, b) = (7, -2)$, we have $d_G(u_2) = 4$. We apply (1.1) at u_2 to get $d_G(u_3) = 6 \notin \{2, a+b-1 = 4\}$, a contradiction. For $(a, b) = (7, -3)$, we have $d_G(u_2) = 3$. We apply (1.1) at u_2 to get $d_G(u_3) = 6 \notin \{2, 4, a+b-1 = 3\}$, a contradiction.

For $(a, b) = (7, -1)$, we have $d_G(u_2) = 5$. We apply (1.1) at u_2 to get $d_G(u_3) = 4$. It implies that $u_3 = v$ and $p = 3$. Together with $t_1' = 3$, we have that $d_G(w_1) = d_G(w_2) = 2$.

Now consider $d_1 \in \{2, a+b-1\}$ and $d_2 \in \{3, 4\}$. Let

$$t_2' = |\{x : d_G(x) = 2,\ x \in \{u_1, u_{p-1}, w_1\}\}|,$$

hence $t_2' = 0, 1, 2, 3$. Together with (3.10) we have

$$2t_2' + (a+b-1)(3-t_2') + d_2 = 4a + b - 16 \qquad (t_2' = 0, 1, 2, 3). \tag{3.11}$$

- $t_2' = 0$. By (3.11), $a - 2b = 13 + d_2$. Since $d_G(u_1) = a+b-1$, we have

$$a - 2b = 13 + d_2 \quad \text{ and } \quad d_G(u_2) = -ab - b^2 + 2b - 2$$

with $d_G(u_2) \in \{2, a+b-1\}$, which, respectively, implies no integer solution since $d_2 \in \{3, 4\}$, a contradiction.

- $t_2' = 1$. By (3.11), $2a - b = 16 + d_2$. First consider $d_G(u_1) = a+b-1$, then we have

$$2a - b = 16 + d_2 \quad \text{ and } \quad d_G(u_2) = -ab - b^2 + 2b - 2$$

with $d_G(u_2) \in \{2, a+b-1\}$, which, respectively, implies no integer solution since $d_2 \in \{3,4\}$, a contradiction. Now consider $d_G(u_1) = 2$, then we have

$$2a - b = 16 + d_2 \quad \text{ and } \quad d_G(u_2) = 2a + b - 8,$$

with $d_G(u_2) \in \{2, a+b-1\}$, which, respectively, implies no integer solution since $d_2 \in \{3,4\}$, a contradiction.

- $t_2' = 2$. By (3.11), $3a = 19 + d_2$. Since a is an integer, we have $d_2 \notin \{3,4\}$, a contradiction.
- $t_2' = 3$. By (3.11), $4a + b = 22 + d_2$. Since $d_G(u_1) = 2$, we have

$$4a + b = 22 + d_2 \quad \text{ and } \quad d_G(u_2) = 2a + b - 8$$

with $d_G(u_2) \in \{2, a+b-1\}$. For $d_G(u_2) = 2$, the equation system implies no integer solution such that $a+b-1 \geq 3$, a contradiction; for $d_G(u_2) = a+b-1$, the equation system implies that $a = 7$, then $a+b-1 = 4-d_2$. If $d_2 = 3$, then $(a,b) = (7,-3)$ and $d_G(u_2) = 3$. Applying (1.1) at u_2, we get that $d_G(u_3) = 6 \notin \{2, a+b-1 = 3\}$, a contradiction. If $d_2 = 4$, then $(a,b) = (7,-2)$ and $d_G(u_2) = 4$. We apply (1.1) at u_2 to get that $d_G(u_3) = 6 \notin \{2, a+b-1 = 4\}$, a contradiction.

Finally, we consider $d_1, d_2 \in \{3,4\}$. Moreover, by Fig. 1, we have $d_1 = d_2 = 3$ and $\widetilde{G} \cong T_7, T_8$. Let

$$t_2'' = |\{x : d_G(x) = 2,\, x \in \{u_1, u_{p-1}\}\}|,$$

hence $t_2'' = 0, 1, 2$. Combining with (3.10) we have

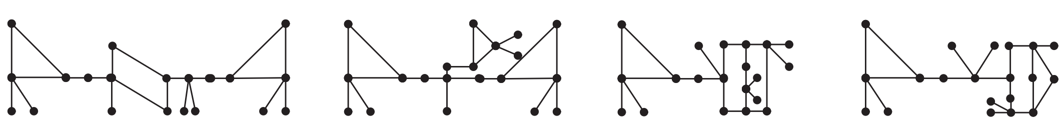
$$2t_2'' + (a+b-1)(2-t_2'') + 3 + 3 = 4a + b - 16 \quad (t_2'' = 0, 1, 2). \tag{3.12}$$

If $t_2'' = 0$, then (3.12) gives $2a - b = 20$. Since $d_G(u_1) = a+b-1$, we have $d_G(u_2) = -ab - b^2 + 2b - 2 \in \{2, a+b-1\}$, which gives no integer solution satisfying $a+b-1 \geq 3$, a contradiction.

If $t_2'' = 1$, then (3.12) gives $a = \frac{23}{3}$, a contradiction.

If $t_2'' = 2$, then (3.12) gives $4a + b = 26$. Since $d_G(u_1) = 2$, we have $d_G(u_2) = 2a + b - 8 \in \{2, a+b-1\}$. For $d_G(u_2) = 2$, there is no integer solution such that $a+b-1 \geq 3$, a contradiction. So $d_G(u_2) = a+b-1$, which gives that $(a,b) = (7,-2)$ and $d_G(u_2) = 4$. We apply (1.1) at u_2 to get $d_G(u_3) = 6 \notin \{2, a+b-1 = 4\}$, a contradiction.

This completes the proof. $\square$



$G_{28} \in \mathscr{G}_{6,-1}$ $G_{29} \in \mathscr{G}_{6,-1}$ $G_{30} \in \mathscr{G}_{6,-1}$ $G_{31} \in \mathscr{G}_{6,-1}$

Figure 3: Graphs G_{28}, G_{29}, G_{30} and G_{31}.

Proposition 1. *Let $G \in \mathscr{G}_{a,b}$ be a tricyclic graph with pendants. If $\widetilde{G}$ contains an internal cycle $C_p = ww_1 \ldots w_{p-1}w$ with $d_{\widetilde{G}}(w) = 3$ or 4, then $G \cong G_{28} \in \mathscr{G}_{6,-1}$, or $G \cong G_{29} \in \mathscr{G}_{6,-1}$, or $G \cong G_{30} \in \mathscr{G}_{6,-1}$, or $G \cong G_{31} \in \mathscr{G}_{6,-1}$; see Fig. 3.*

Proof. Based on Fig. 1, we obtain that $G \in \{T_1, T_2, T_4, T_5, T_6, T_7, T_8, T_{10}, T_{11}\}$. By Lemma 3.1 and Lemma 3.2, it's easy to see that $G \neq T_2, T_7$.

Case 1. $G \in \mathscr{T}_n^3$. In this case, $\widetilde{G}$ contains three cycles, say C_{r_1}, C_{r_2} and C_{r_3}. Then $G \in \{T_1, T_4, T_5, T_6\}$. By Lemma 3.1 and Lemma 3.2, if C_{r_i} is an internal cycle of $\widetilde{G}$, then $r_i = 3$.

• $G = T_1$; see Fig. 1. Then $r_1 = r_3 = 3$. Note that $d_{\widetilde{G}}(u) = d_{\widetilde{G}}(v) = 4$. By Lemma 3.2, we obtain that $G \in \mathscr{G}_{7,-1}$ and $d_G(u) = d_G(v) = 4$. Denote $C_{r_1} = ux_1x_2u$. Suppose C_{r_2} consists of the two internal paths of $\widetilde{G}$: $P_{k_1+1} = us_1s_2 \ldots s_{k_1}(s_{k_1} = v)$, $P_{k_2+1} = ut_1t_2 \ldots t_{k_2}(t_{k_2} = v)$ connects u, v. Then Lemma 3.2 implies that $d_G(x_1) = d_G(s_1) = d_G(t_1) = 2, d_G(x_2) = a + b - 1 = 5$. Applying (1.1) at s_1 and t_1 respectively yields $d_G(s_2) = d_G(t_2) = 5$. Lemma 3.2 implies that $d_G(v) = 4$, hence, $s_2, t_2 \neq v$. We apply (1.1) at s_2 and t_2 respectively to get $d_G(s_3) = d_G(t_3) = 4$, which implies that $s_3 = t_3 = v$. However, by Lemma 3.2 we have $d_G(s_2) = d_G(t_2) = 2$, a contradiction.

• $G = T_4$; see Fig. 1. Then $r_1 = r_3 = 3$. Note that $d_{\widetilde{G}}(u_2) = d_{\widetilde{G}}(v_2) = 3$. By Lemma 3.1, we obtain that $G \in \mathscr{G}_{6,-1}$ and $d_G(u) = d_G(v) = 3$. Denote $C_{r_1} = u_2x_1x_2u_2$, $C_{r_3} = v_2y_1y_2v_2$. Suppose the internal paths of $\widetilde{G}$: $P_{k_1+1} = u_2q_1q_2 \ldots q_{k_1}(q_{k_1} = u_1)$ connects u_2, u_1, $P_{k_2+1} = u_1s_1s_2 \ldots s_{k_2}(s_{k_2} = v_1)$ and $P_{k_3+1} = u_1t_1t_2 \ldots t_{k_3}(t_{k_3} = v_1)$ connects u_1, v_1, $P_{k_4+1} = v_1z_1z_2 \ldots z_{k_4}(z_{k_4} = v_2)$ connects v_1, v_2. Then Lemma 3.1 implies that $d_G(x_1) = d_G(y_1) = d_G(q_1) = 2$, $d_G(x_2) = d_G(y_2) = a + b - 1 = 4$. Applying (1.1) at q_1 yields $d_G(q_2) = 4$.

We first assume that $q_2 = u_1$. We apply (1.1) at u_1 to get $d_G(s_1) + d_G(t_1) = 4$, which implies that $d_G(s_1) = d_G(t_1) = 2$. Applying (1.1) at s_1 and t_1 respectively yields $d_G(s_2) = d_G(t_2) = 3$. Hence, we have $s_2 = t_2 = v_1$ with $d_G(v_1) = 3$. Applying (1.1) at v_1, we get that $d_G(z_1) = 4$, moreover, $d_G(z_2) = 2$, $d_G(z_3) = 3$. Therefore, $z_3 = v$. Thus, it's easy to check that $G \cong G_{28} \in \mathscr{G}_{6,-1}$; see Fig. 3.

Now we assume $q_2 \neq u_1$. By the similar proof as in the case of $q_2 = u_1$, we may also get the graph $G \cong G_{28} \in \mathscr{G}_{6,-1}$.

• $G = T_5$; see Fig. 1. Then $r_3 = 3$. Note that $d_{\widetilde{G}}(v) = 3$. By Lemma 3.1, we obtain that $G \in \mathscr{G}_{6,-1}$ and $d_G(v) = 3$. Denote $C_{r_1} = vx_1x_2v$. Suppose the internal paths of $\widetilde{G}$: $P_{k_1+1} = vy_1y_2 \ldots y_{k_1}(y_{k_1} = u)$ connects v, u. Then Lemma 3.1 implies that $d_G(x_1) = d_G(y_1) = 2$, $d_G(x_2) = a + b - 1 = 4$. Applying (1.1) at y_1 yields $d_G(y_2) = 4$, moreover, $d_G(y_3) = 3 \notin \{2, 5, a + b - 1 = 4\}$, a contradiction.

• $G = T_6$; see Fig. 1. Then $r_1 = r_2 = r_3 = 3$. Note that $d_{\widetilde{G}}(v_1) = d_{\widetilde{G}}(v_2) = d_{\widetilde{G}}(v_3) = 3$. By Lemma 3.1, we obtain that $G \in \mathscr{G}_{6,-1}$ and $d_G(v_1) = d_G(v_2) = d_G(v_3) = 3$. Denote $C_{r_1} = v_1x_1x_2v_1$, $C_{r_2} = v_2y_1y_2v_2$, $C_{r_3} = v_3z_1z_2v_3$. Suppose the internal paths of $\widetilde{G}$: $P_{k_1+1} = v_1s_1s_2 \ldots s_{k_1}(s_{k_1} = u)$ connects v_1, u, $P_{k_2+1} = ut_1t_2 \ldots t_{k_2}(t_{k_2} = v_2)$ connects u, v_2, $P_{k_3+1} = uq_1q_2 \ldots q_{k_3}(q_{k_3} = v_3)$ connects u, v_3. Then Lemma 3.1 implies that $d_G(x_1) = d_G(y_1) = d_G(z_1) = d_G(s_1) = 2$, $d_G(x_2) = d_G(y_2) = d_G(z_2) = a + b - 1 = 4$. Applying (1.1) at s_1 yields $d_G(s_2) = 4$.

We first assume that $s_2 = u$. Then we have $d_G(u) = 4$. We apply (1.1) at u to get that $d_G(t_1) + d_G(q_1) = 4$, which implies $d_G(t_1) = d_G(q_1) = 2$. Applying (1.1) at t_1 and q_1 respectively yields $d_G(t_2) = d_G(q_2) = 3$. Hence, we have $t_2 = v_2$, $q_2 = v_3$. Thus, it's easy to check that $G \cong G_{29} \in \mathscr{G}_{6,-1}$; see Fig. 3.

Now we assume $s_2 \neq u$. Applying (1.1) at s_2, we have $d_G(s_3) = 3$, which implies that $s_3 = u$ with $d_G(u) = 3$. Applying (1.1) at u yields $d_G(t_1) + d_G(q_1) = 4$. Hence, $d_G(t_1) = d_G(q_1) = 2$. We apply (1.1) at t_1 to get that $d_G(t_2) = 4$, moreover, $d_G(t_3) = 3$. Therefore, $t_3 = v_2$. However, Lemma 3.1 implies that $d_G(t_2) = 2$, a contradiction.

Case 2. $G \in \mathscr{T}_n^4$. In this case, $\widetilde{G}$ contains an internal cycle. Then $G \in \{T_8, T_{10}, T_{11}$. By Lemma 3.1 and Lemma 3.2, if C_{r_i} is an internal cycle of $\widetilde{G}$, then $r_i = 3$.

• $G = T_8$; see Fig. 1. Then $r_1 = 3$. Note that $d_{\widetilde{G}}(u) = 4$. By Lemma 3.2, we obtain that $G \in \mathscr{G}_{7,-1}$ and $d_G(u) = 4$. Denote $C_{r_1} = ux_1x_2u$. Suppose the internal paths of $\widetilde{G}$: $P_{k_1+1} = us_1s_2 \ldots s_{k_1}(s_{k_1} = v_1)$ connects u, v_1, $P_{k_2+1} = ut_1t_2 \ldots t_{k_2}(t_{k_2} = v_2)$ connects u, v_2, $P_{k_3+1} = v_1y_1y_2 \ldots y_{k_3}(y_{k_3} = v_2)$, $P_{k_4+1} = v_1z_1z_2 \ldots z_{k_4}(z_{k_4} = v_2)$ connects v_1, v_2. Then Lemma 3.2 implies that $d_G(x_1) = d_G(s_1) = d_G(t_1) = 2$, $d_G(x_2) = a + b - 1 = 5$. We apply (1.1) at s_1 and t_1 respectively to get $d_G(s_2) = d_G(t_2) = 5$. If $s_2 \neq v_1$, we apply (1.1) at s_2 to get $d_G(s_3) = 4 \notin \{2, 3, a + b - 1 = 5\}$, a contradiction. Hence, $s_2 = v_1$. Similarly, $t_2 = v_2$. Applying (1.1) at v_1 yields $d_G(y_1) + d_G(z_1) = 5$, it's easy to see that this is impossible.

• $G = T_{10}$; see Fig. 1. Then $r_1 = 3$. Note that $d_{\widetilde{G}}(u) = 3$. By Lemma 3.1, we obtain that $G \in \mathscr{G}_{6,-1}$ and $d_G(u) = 3$. Denote $C_{r_1} = ux_1x_2u$. Suppose the internal paths of $\widetilde{G}$: $P_{k_1+1} = uy_1y_2\dots y_{k_1}(y_{k_1} = v)$ connects u, v. Then Lemma 3.1 implies that $d_G(x_1) = d_G(y_1) = 2$, $d_G(x_2) = a + b - 1 = 4$. We apply (1.1) at y_1 to get $d_G(y_2) = 4$. If $y_2 \neq v$, we apply (1.1) at y_2 to get $d_G(y_3) = 3 \notin \{2, a+b-1 = 4\}$, a contradiction. Hence, $y_2 = v$ with $d_G(v) = 4$. Denote $N_G(v) = \{y_1, s_1, s_2, s_3\}$, then applying (1.1) at v yields $d_G(s_1) + d_G(s_2) + d_G(s_3) = 5$, it's easy to see that this is impossible.

• $G = T_{11}$; see Fig. 1. Then $r_1 = 3$. Note that $d_{\widetilde{G}}(u_1) = 3$. By Lemma 3.1, we obtain that $G \in \mathscr{G}_{6,-1}$ and $d_G(u_1) = 3$. Denote $C_{r_1} = u_1x_1x_2u_1$. Suppose the internal paths of $\widetilde{G}$: $P_{k_1+1} = u_1y_1y_2\dots y_{k_1}(y_{k_1} = u_2)$ connects u_1, u_2, $P_{k_2+1} = u_2s_1s_2\dots s_{k_2}(s_{k_2} = v_1)$ connects u_2, v_1, $P_{k_3+1} = u_2t_1t_2\dots t_{k_3}(t_{k_3} = v_2)$ connects u_2, v_2, $P_{k_4+1} = v_1z_1z_2\dots z_{k_4}(z_{k_4} = v_2)$, $P_{k_5+1} = v_1q_1q_2\dots q_{k_5}(q_{k_5} = v_2)$ connects v_1, v_2. Then Lemma 3.1 implies that $d_G(x_1) = d_G(y_1) = 2$, $d_G(x_2) = a + b - 1 = 4$. Applying (1.1) at y_1, yields $d_G(y_2) = 4$.

We first consider $y_2 = u_2$ with $d_G(u_2) = 4$. Applying (1.1) at u_2, we have $d_G(s_1) + d_G(t_1) = 4$, which implies that $d_G(s_1) = d_G(t_1) = 2$. We apply (1.1) at s_1 and t_1 respectively to get $d_G(s_2) = d_G(t_2) = 3$. It implies that $s_2 = v_1$, $t_2 = v_2$ with $d_G(v_1) = d_G(v_2) = 3$. Applying (1.1) at v_1, yields $d_G(z_1) + d_G(q_1) = 6$. Hence, without loss of generality, we assume $d_G(z_1) = 2$, $d_G(q_1) = 4$. Therefore, $d_G(z_2) = 4$, $d_G(z_3) = 3$; $d_G(q_2) = 2$, $d_G(q_3) = 3$. Thus, $z_3 = q_3 = v_2$. It's easy to check that $G \cong G_{30} \in \mathscr{G}_{6,-1}$; see Fig. 3.

Now we consider $y_2 \neq u_2$. Then we apply (1.1) at y_2 to get $d_G(y_3) = 3$, which implies that $y_3 = u_2$ with $d_G(u_2) = 3$. Applying (1.1) at u_2 yields $d_G(s_1) + d_G(t_1) = 4$. Hence, $d_G(s_1) = d_G(t_1) = 2$. Applying (1.1) at s_1 and t_1 respectively, we have $d_G(s_2) = d_G(t_2) = 4$. For the subcase of $s_2 = v_1$ with $d_G(v_1) = 4$, we may get $d_G(z_1) + d_G(q_1) = 4$ by applying (1.1) at v_1. Hence, we have $d_G(z_1) = d_G(q_1) = 2$. We apply (1.1) at z_1 and q_1 respectively to get $d_G(z_2) = d_G(q_2) = 3$. Therefore, $z_2 = q_2 = v_2$ with $d_G(v_2) = 3$. It implies that $t_2 \neq v_2$. Then applying (1.1) at t_2 yields $d_G(t_3) = 3$. Thus, $t_3 = v_2$. It's easy to check that $G \cong G_{31} \in \mathscr{G}_{6,-1}$; see Fig. 3. For the subcase of $s_2 \neq v_1$, we may also get the graph $G \cong G_{31} \in \mathscr{G}_{6,-1}$ by the similar proof as above.

Thus, we complete the proof. $\square$

Proposition 2. *Let G be a tricyclic graph with pendants satisfying $\widetilde{G} = T_3$; see Fig. 1. Then $G \notin \mathscr{G}_{a,b}$.*

Proof. For the graph G, let u be the common vertex of C_{r_1}, C_{r_2} and C_{r_3} as depicted in T_3 of Fig. 1. Assume that $N_{\widetilde{G}}(u) \cap V_{C_{r_1}} = \{u_1, u_{p-1}\}, N_{\widetilde{G}}(u) \cap V_{C_{r_2}} = \{w_1, w_2\}, N_{\widetilde{G}}(u) \cap V_{C_{r_3}} = \{w_3, w_4\}$. Suppose to the contrary that $G \in \mathscr{G}_{a,b}$. Note that $d_G(u_1), d_G(u_{p-1}), d_G(w_1), d_G(w_2), d_G(w_3), d_G(w_4) \in \{2, a+b-1\}$. Then we may let

$$t = |x : d_G(x) = 2, x \in \{u_1, u_{p-1}, w_1, w_2, w_3, w_4\}|. \tag{3.13}$$

Hence, $0 \le t \le 6$. Note that if $t = 0$, we have $d_G(u_1) = a + b - 1$; if $t \in \{1, 2, 3, 4, 5, 6\}$, without loss of generality, we can assume $d_G(u_1) = 2$. We proceed by distinguish the following two possible cases. Note that

$$d_G(u_2) \in \{2, \quad a + b - 1\}. \tag{3.14}$$

Case 1. $d_G(u) = a + b - 1 > 6$. In this case as $b \le -1$, we have $a \ge 9$.

If $d_G(u_1) = 2$, applying (1.1) at u_1, we have $d_G(u_2) = a - 3 \in \{2, a+b-1\}$. As $a \ge 9$, we get that $a - 3 \neq 2$. Thus, $d_G(u_2) = a - 3 = a + b - 1$, i.e., $b = -2$; if $d_G(u_1) = a + b - 1$, applying (1.1) at u_1, we have $d_G(u_2) = -ab - b^2 - a + b + 3$. Apply Lemma 1.8(ii) at u yields

$$d_G(u_1) + d_G(u_{p-1}) + d_G(w_1) + d_G(w_2) + d_G(w_3) + d_G(w_4) = -ab - b^2 + 2b + 6.$$

Together with (3.13) we have $2t + (a + b - 1)(6 - t) = -ab - b^2 + 2b + 6$. For $d_G(u_1) = 2$, we have

$$\begin{cases} 2t + (a + b - 1)(6 - t) = -ab - b^2 + 2b + 6 \qquad (t = 1, 2, 3, 4, 5, 6) \\ b = -2 \end{cases}$$

For $d_G(u_1)=a+b-1$, we have

$$\begin{cases} 2t+(a+b-1)(6-t)=-ab-b^2+2b+6 \qquad (t=0) \\ d_G(u_2)=-ab-b^2-a+b+3 \end{cases}$$

It's routine to check that there is no integer solution satisfying $a+b-1>6$, a contradiction.

Case 2. $d_G(u)=6$. In this case, $a+b-1\geq 3$

If $d_G(u_1)=2$, applying (1.1) at u_1, we have $d_G(u_2)=2a+b-10$; if $d_G(u_1)=a+b-1$, applying (1.1) at u_1, we have $d_G(u_2)=-ab-b^2+2b-4$.

Applying (1.1) at u yields

$$d_G(u_1)+d_G(u_{p-1})+d_G(w_1)+d_G(w_2)+d_G(w_3)+d_G(w_4)=6a+b-36.$$

Together with (3.24) we have $2t+(a+b-1)(6-t)=6a+b-36$. For $d_G(u_1)=2$, we have

$$\begin{cases} 2t+(a+b-1)(6-t)=6a+b-36 \qquad (t=1,2,3,4,5,6) \\ d_G(u_2)=2a+b-10 \end{cases} \tag{3.15}$$

For $d_G(u_1)=a+b-1$, we have

$$\begin{cases} 2t+(a+b-1)(6-t)=6a+b-36 \qquad (t=0) \\ d_G(u_2)=-ab-b^2+2b-4 \end{cases} \tag{3.16}$$

In view of (3.14) and (3.15), if $d_G(u_2)=2$, then we obtain that $(a-9)(10-t)=0$. It implies that $a=9$. Then $b=-6$, hence $a+b-1=2<3$, a contradiction; if $d_G(u_2)=a+b-1$, then $a=9$, so we get that $(5-t)(b+6)=0$, which implies that $t=5$ as $b\neq -6$. Thus $b\in\{-1,-2,-3,-4,-5\}$ since $b\leq -1$ and $a+b-1\geq 3$.

Note that $t=5$, without loss of generality, we assume $d_G(u_1)=d_G(u_{p-1})=2$. If $b=-1$, then $(a,b)=(9,-1)$ and $d_G(u_2)=7$. Applying (1.1) at u_2, it's routine to verify that $d_G(u_3)=6\notin\{d_{\widetilde{G}}(u_3)=2, a+b-1=7\}$, a contradiction; for $b\in\{-2,-3,-4,-5\}$, we will also get a contradiction by the similar proof as above.

From (3.16), we get that $(a,b)=(9,-6)$, hence $a+b-1=2<3$, a contradiction. $\square$

Proposition 3. *Let G be a tricyclic graph with pendants. If $\widetilde{G}=T_9$ (see Fig. 1), then $G\notin\mathscr{G}_{a,b}$.*

Proof. Note that $\widetilde{G}=T_9$, hence $\widetilde{G}$ contains an internal cycle $C_{r_1}: uu_1\ldots u_{p-1}u$ and three paths $P_{k_1+1}: ux_1\ldots x_{k_1}(x_{k_1}=v), P_{k_2+1}: uy_1\ldots y_{k_2}(y_{k_2}=v), P_{k_3+1}: uz_1\ldots z_{k_3}(z_{k_3}=v)$ connecting u,v; see T_9 of Fig. 1. Without loss of generality, we assume $k_2\geq 2, k_3\geq 2$. Thus, $d_G(u_1), d_G(u_{p-1}), d_G(y_1), d_G(z_1)\in\{2,a+b-1\}$. Denote $N_{\widetilde{G}}(u)=\{u_1,u_{p-1},w,y_1,z_1\}$, where $w=v$ if $k_1=1$; $w=x_1\neq v$ if $k_1\geq 2$. Thus, we let $r=d_G(w)\in\{2,3,a+b-1\}$ for convenience. Let

$$t=|\{x: d_G(x)=2, x\in\{u_1,u_{p-1},y_1,z_1\}\ \}|, \tag{3.17}$$

hence, $0\leq t\leq 4$. Note that if $t=0,1$, let $d_G(u_1)=a+b-1$; if $t=2$, we have $d_G(u_1)\in\{2,a+b-1\}$; if $t\in\{3,4\}$, let $d_G(u_1)=2$.

In what follows, according to the vertex u of maximum degree in $\widetilde{G}$, we distinguish the following cases to prove our result.

Case 1. u is an attached vertex, i.e. $d_G(u)=a+b-1>5$. In this case, as $b\leq -1$, we have $a\geq 8$.

If $d_G(u_1)=2$, we apply (1.1) at u_1 to get $d_G(u_2)=a-3\in\{2,a+b-1\}$. Since $a\geq 8$, we get that $a-3\neq 2$. Hence, $d_G(u_2)=a-3=a+b-1$, i.e., $b=-2$; if $d_G(u_1)=a+b-1$, we apply (1.1) at u_1 to get $d_G(u_2)=-ab-b^2-a+b+3$.

Applying Lemma 1.8(ii) at u, we have

$$\sum_{w\in N_{\widetilde{G}}(u)} = -ab - b^2 + 2b + 5. \tag{3.18}$$

Together with (3.17), we have $r + 2t + (a+b-1)(4-t) = -ab - b^2 + 2b + 5$. For $d_G(u_1) = 2$, we have

$$\begin{cases} r + 2t + (a+b-1)(4-t) = -ab - b^2 + 2b + 5 \quad (t = 2,3,4) \\ b = -2 \end{cases} \tag{3.19}$$

For $d_G(u_1) = a+b-1$, we have

$$\begin{cases} r + 2t + (a+b-1)(4-t) = -ab - b^2 + 2b + 5 \quad (t = 0,1,2) \\ d_G(u_2) = -ab - b^2 - a + b + 3 \end{cases} \tag{3.20}$$

Note that $d_G(u_2) \in \{2, a+b-1\}$. In view of (3.19), since a and b are bother integers satisfying $a+b-1 > 5$, it's routine to check that only $r = 3$ with $t = 3$ holds. It implies that $(a,b) = (9,-2)$. Hence, $d_G(u_2) = a+b-1 = 6$. Applying (1.1) at u_2 yields $d_G(u_3) = 10 \notin \{2, a+b-1 = 6\}$, a contradiction. In view (3.20), we get no integer solution satisfying $a+b-1 > 5$, a contradiction.

Case 2. u is a non-attached vertex, i.e. $d_G(u) = 5$. In this case, we have $a+b-1 \geq 3$.

If $d_G(u_1) = a+b-1$, then we apply (1.1) at u_1 to get $d_G(u_2) = -ab - b^2 + 2b - 3 \in \{2, a+b-1\}$. Note that for $d_G(u_2) = 2$, we get $-b(a+b-2) = 5$. As $a+b-1 \geq 3$ and b is an integer, we have $b = -1, a+b-2 = 5$, i.e., $(a,b) = (8,-1)$. If $d_G(u_1) = 2$, then we apply (1.1) at u_1 to get $d_G(u_2) = 2a + b - 9$. Note that

$$d_G(u_2) \in \{2, a+b-1\}. \tag{3.21}$$

Applying (1.1) at u, we have

$$\sum_{w\in N_G(u)} d_G(w) = 5a + b - 25. \tag{3.22}$$

Together with (3.17), we have $r + 2t + (a+b-1)(4-t) = 5a + b - 25$. For $d_G(u_1) = 2$, we have

$$\begin{cases} r + 2t + (a+b-1)(4-t) = 5a + b - 25 \quad (t = 2,3,4) \\ d_G(u_2) = 2a + b - 9 \end{cases} \tag{3.23}$$

For $d_G(u_1) = a+b-1$, we have

$$\begin{cases} r + 2t + (a+b-1)(4-t) = 5a + b - 25 \quad (t = 0,1,2) \\ d_G(u_2) = -ab - b^2 + 2b - 3 \end{cases} \tag{3.24}$$

In view of (3.21) and (3.23), note that a and b are both integers satisfying $a+b-1 \geq 3$. It's routine to check that only $d_G(u_2) = a+b-1$ with $r \in \{3, a+b-1\}$ holds. If $r = 3$, we have $(a,b) = (8,-4)$. Then $d_G(u_2) = a+b-1 = 3$. Applying (1.1) at u_2, we have $d_G(u_3) = 8 \notin \{2, 5, a+b-1 = 3\}$, a contradiction. If $r = a+b-1$, we have $a = 8$ and $(b+5)(t-4) = 0$, which implies that $t = 4$, thus $b \in \{-1,-2,-3,-4\}$ since $a+b-1 \geq 3$ and $b \leq -1$. For $b = -1$, we have $(a,b) = (8,-1)$. Since $d_G(u_1) = 2, d_G(u_2) = 6$, applying (1.1) at u_2, it's simple to verify that $d_G(u_3) = 5 \notin \{d_{\widetilde{G}}(u_3) = 2, a+b-1 = 6\}$, it's a contradiction; for $b \in \{-2,-3,-4\}$, we will also get a contradiction by the similar proof as above.

In view of (3.21) and (3.24), if $d_G(u_2) = 2$, then $(a,b) = (8,-1)$. For $r = 2$, we have $t = 3$; for $r = 3$, we have $t = \frac{13}{4}$; for $r = a+b-1$, we have $t = 4$. Each is impossible since $t \in \{0,1,2\}$. If $d_G(u_2) = a+b-1$, it's routine to check that there is no integer solution such that $a+b-1 \geq 3$ since $r \in \{2, 3, a+b-1\}$ and $t = 0,1,2$. □

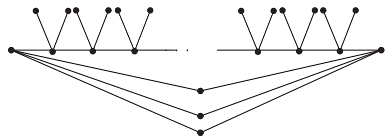


Figure 4: Graphs $G_{32} \in \mathscr{G}_{7,-2}$.

Proposition 4. *Let $G \in \mathscr{G}_{a,b}$ be a tricyclic graph with pendants with $\widetilde{G} = T_{12}$. Then $G \cong G_{32} \in \mathscr{G}_{7,-2}$, where G_{29} is depicted in Fig. 4.*

Proof. Note that $\widetilde{G} = T_{12}$ (see Fig. 1), hence $\widetilde{G}$ consists of four internal paths connecting u and v: $P_{k_1+1} = us_1 \dots s_{k_1}(s_{k_1} = v), P_{k_2+1} = ut_1 \dots t_{k_2}(t_{k_2} = v), P_{k_3+1} = ux_1 \dots x_{k_3}(x_{k_3} = v), P_{k_4+1} = uy_1 \dots y_{k_4}(y_{k_4} = v)$. Without loss of generality, we assume $k_4 \geq k_3 \geq k_2 \geq k_1$. If $k_1 = 1$, then $N_{\widetilde{G}}(u) = \{v, t_1, x_1, y_1\}$; if $k_1 \geq 2$, then $N_{\widetilde{G}}(u) = \{s_1, t_1, x_1, y_1\}$ with $s_1 \neq v$. Obviously, $k_4 \geq k_3 \geq k_2 \geq 2$, then $d_G(t_1), d_G(x_1), d_G(y_1) \in \{2, a+b-1\}$. Let

$$t = |\{x : d_G(x) = 2, \quad x \in \{t_1, x_1, y_1\}\}|, \tag{3.25}$$

hence $0 \leq t \leq 3$. Denote $N_{\widetilde{G}}(u) = \{w, t_1, x_1, y_1\}$, where $w = v$ if $k_1 = 1$; $w = s_1 \neq v$ if $k_1 \geq 2$. Thus, $d_G(w) = r \in \{2, 4, a+b-1\}$.

Case 1. $d_G(u) = a+b-1 > 4$. In this case, since $b \leq -1$, we have $a \geq 7$.

If $d_G(t_1) = 2$, applying (1.1) at t_1, we have $d_G(t_2) = a-3 \in \{2, 4, a+b-1\}$. Since $a \geq 7$, we get that $a-3 \neq 2$. For $d_G(t_2) = 4$, we get $(a,b) = (7,-1)$; for $d_G(t_2) = a+b-1$, we get $b = -2$. If $d_G(t_1) = a+b-1$, applying (1.1) at t_1, we have $d_G(t_2) = -ab - b^2 - a + b + 3$.

Applying (1.1) at u yields

$$d_G(w) + d_G(t_1) + d_G(x_1) + d_G(y_1) = -ab - b^2 + 2b + 4. \tag{3.26}$$

By (3.25) we have

$$r + 2t + (a+b-1)(3-t) = -ab - b^2 + 2b + 4, \qquad (t = 0,1,2,3) \tag{3.27}$$

- $t = 0$. (3.27) gives $ab + b^2 + 3a + b = 7 - r$. Since $d_G(t_1) = a+b-1$, we have

$$ab + b^2 + 3a + b = 7 - r \quad \text{and} \quad d_G(t_2) = -ab - b^2 - a + b + 3$$

with $d_G(t_2) \in \{2, 4, a+b-1\}$. It's routine to check that there is no integer solution such that $a+b-1 > 4$, a contradiction.

- $t = 1$. (3.27) gives $ab+b^2+2a = 4-r$. Without loss of generality, we assume $d_G(t_1) = 2$, then $(a,b) = (7,-1)$ or $b = -2$. It's easy to check that both are impossible since $r \in \{2, 4, a+b-1\}$.
- $t = 2$. (3.27) gives $ab + b^2 + a - b = 1 - r$. Without loss of generality, we assume $d_G(t_1) = 2$, then $(a,b) = (7,-1)$ or $b = -2$. If $(a,b) = (7,-1)$, then $r = -1$, a contradiction; if $b = -2$ with $d_G(t_2) = a+b-1$, then $a = 5+r$. It's easy to see that only $r = 4$ holds. So we have $v \in N_G(u)$ and $d_G(v) = 4$, moreover, $(a,b) = (9,-2)$ and $d_G(t_2) = 6$. Applying (1.1) at t_2, we have $d_G(t_3) = 10 \notin \{2, 4, a+b-1 = 6\}$, a contradiction.
- $t = 3$. (3.27) gives $ab + b^2 - 2b + 2 + r = 0$. Note that $d_G(t_1) = 2$, hence $(a,b) = (7,-1)$ or $b = -2$.

If $(a,b) = (7,-1)$, then $r = 2$. Therefore, we have $d_G(s_1) = d_G(t_1) = d_G(x_1) = d_G(y_1) = 2$ together with $t = 3$. Applying (1.1) at s_1 yields $d_G(s_2) = 4$, which implies $s_2 = v$ with $d_G(v) = 4$. Similarly, $t_2 = x_2 = y_2 = v$. Thus, $N_G(v) = \{s_1, t_1, x_1, y_1\}$. We may check that this is impossible by applying (1.1) at v.

If $b = -2$ with $d_G(t_2) = a+b-1$, then $2a = 10+r$. Since $a+b-1 > 4$ and a is an integer, it's easy to verify that $r \notin \{2, 4, a+b-1\}$, a contradiction.

Case 2. $d_G(u) = 4$. In this case, $a + b - 1 \geq 3$.

If $d_G(t_1) = 2$, applying (1.1) at t_1, we have $d_G(t_2) = 2a + b - 8$. If $d_G(t_1) = a + b - 1$, applying (1.1) at t_1, we have $d_G(t_2) = -ab - b^2 + 2b - 2$.

Applying (1.1) at u yields

$$d_G(w) + d_G(t_1) + d_G(x_1) + d_G(y_1) = 4a + b - 16 \tag{3.28}$$

From (3.25) we have

$$r + 2t + (a + b - 1)(3 - t) = 4a + b - 16, \qquad (t = 0, 1, 2, 3.) \tag{3.29}$$

• $t = 0$. (3.29) gives $a - 2b = 13 + r$. Since $d_G(t_1) = a + b - 1$, we have

$$a - 2b = 13 + r \quad \text{and} \quad d_G(t_2) = -ab - b^2 + 2b - 2.$$

Note that $d_G(t_2) \in \{2, 4, a+b-1\}$. It's routine to check that there is no integer solution such that $a+b-1 \geq 3$ since $r \in \{2, 4, a + b - 1\}$.

• $t = 1$. (3.29) gives $2a - b = 16 + r$. Without loss of generality, we assume $d_G(t_1) = a + b - 1$, then

$$2a - b = 16 + r \quad \text{and} \quad d_G(t_2) = -ab - b^2 + 2b - 2.$$

Note that $d_G(t_2) \in \{2, 4, a+b-1\}$. It's routine to check that there is no integer solution such that $a+b-1 \geq 3$ since $r \in \{2, 4, a + b - 1\}$.

• $t = 2$. (3.29) gives that $3a = 19 + r$. Since a is an integer, $r \neq 4$. Hence, $r \in \{2, a + b - 1\}$. Without loss of generality, we assume $d_G(t_1) = d_G(x_1) = 2, d_G(y_1) = a + b - 1$, then

$$3a = 19 + r \quad \text{and} \quad d_G(t_2) = 2a + b - 8.$$

Note that $d_G(t_2) \in \{2, 4, a + b - 1\}$. It's routine to check that only $r = 2$ holds, which implies that $a = 7$, moreover, $d_G(s_1) = 2$. For $d_G(t_2) = 2$, we have $b = -4$. However, $a + b - 1 = 2 < 3$, a contradiction. For $d_G(t_2) = 4$, we have $b = -2$. For $d_G(t_2) = a + b - 1$, we have $b \in \{-1, -2, -3\}$.

First we consider $(a, b) = (7, -2)$. In this subcase, $a + b - 1 = 4$ and $d_G(s_2) = d_G(x_2) = d_G(t_2) = 4$. If $\{s_2, t_2, x_2\}$ contains a member, say s_2, such that $s_2 \neq v$, then applying (1.1) at s_2 yields $d_G(s_3) = 6 \notin \{2, 4\}$, a contradiction. Therefore, we get that $s_2 = x_2 = t_2 = v$. Note that $d_G(y_1) = a + b - 1 = 4$. It's easy to see that $y_1 \neq v$ since G has pendant vertices. Applying (1.1) at y_1, we have $d_G(y_2) = 4$. Continue the process, we may finally obtain that $d_G(y_i) = 4$ for all $1 \leq i \leq k_4$, where $k_4 \geq 2$. Thus, we get the graph $G \cong G_{32} \in \mathscr{G}_{7,-2}$; see Fig. 4.

Next, we consider $b \in \{-1, -3\}$.

If $b = -1$, then $(a, b) = (7, -1)$. In this subcase, $d_G(y_1) = a + b - 1 = 5$, and $d_G(s_2) = d_G(t_2) = d_G(x_2) = 5$. If $\{s_2, t_2, x_2\}$ contains a member, say s_2, such that $s_2 \neq v$, then applying (1.1) at s_2 yields $d_G(s_3) = 4$. It implies that $s_3 = v$ with $d_G(v) = 4$. Therefore, $t_2 \neq v, x_2 \neq v$ and $t_3 = x_3 = v$. It's easy to see that this is impossible by applying (1.1) at v. Hence, we have $s_2 = t_2 = x_2 = v$ with $d_G(v) = 5$. Note that $d_G(y_1) = 5$. We apply (1.1) at y_1 to get $d_G(y_2) = 2$, moreover, $d_G(y_3) = 4 \notin \{2, 5\}$, a contradiction.

If $b = -3$, then $(a, b) = (7, -3)$. In this subcase, $d_G(y_1) = a + b - 1 = 3$, and $d_G(s_2) = d_G(t_2) = d_G(x_2) = 3$. Applying (1.1) at s_2 yields $d_G(s_3) = 6 \notin \{2, 4, a + b - 1 = 3\}$, a contradiction.

• $t = 3$. (3.29) gives that $4a + b = 22 + r$. As $d_G(t_1) = 2$, we have

$$4a + b = 22 + r \quad \text{and} \quad d_G(t_2) = 2a + b - 8. \tag{3.30}$$

Note that $d_G(t_2) \in \{2, 4, a + b - 1\}$. If $d_G(t_2) = 2$, then (3.30) gives that $2a = 12 + r$. It's routine to check that there is no integer solution such that $a + b - 1 \geq 3$, a contradiction.

If $d_G(t_2) = 4$, then (3.30) gives that $2a = 10 + r$. For $r = 2$, we have $b = 0$, a contradiction to $b \leq -1$. For $r \in \{4, a + b - 1\}$, we have $(a, b) = (7, -2)$.

If $d_G(t_2) = a + b - 1$, then (3.30) gives that $a = 7$, this implies that $a + b - 1 = r$. Note that $a + b - 1 \geq 3$, then $r \neq 2$. For $r = 4$, we have $(a, b) = (7, -2)$. For $r = a + b - 1$, we have $b \in \{-1, -2, -3\}$ since $b \leq -1$. By a similar proof as in the discussion of $t = 2$, we may also get $G \cong G_{32} \in \mathscr{G}_{7,-2}$.

This completes the proof. □

By a similar discussion as in the proof of Lemma 3.5, we can show the next lemma. We omit its procedure.

Proposition 5. *Let G be a tricyclic graph with pendants. If $\widetilde{G} = T_{13}$ (see Fig. 1), then $G \notin \mathscr{G}_{a,b}$.*

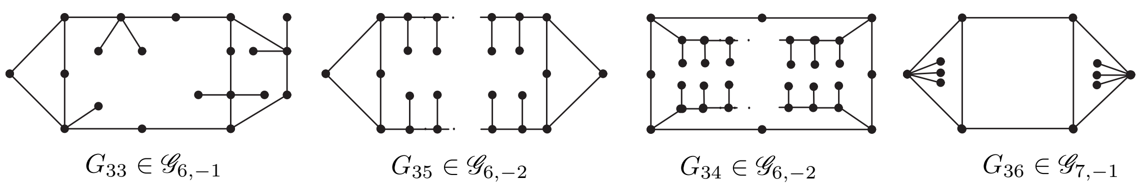


Figure 5: Graphs G_{33}, G_{34}, G_{35}, G_{36}.

Proposition 6. *Let $G \in \mathscr{G}_{a,b}$ be a tricyclic graph with pendants satisfying $\widetilde{G} = T_{14}$. Then $G \cong G_{33} \in \mathscr{G}_{6,-1}$ or, $G \cong G_{34} \in \mathscr{G}_{6,-2}$ or, $G \cong G_{35} \in \mathscr{G}_{6,-2}$ or, $G \cong G_{36} \in \mathscr{G}_{7,-1}$, where G_{33}, G_{34}, G_{35}, G_{36} are depicted in Fig. 5.*

Proof. Note that $\widetilde{G} = T_{14}$; see Fig. 1, hence $\widetilde{G}$ contains two internal paths $P_{k_1+1} = u_1 s_1 s_2 \ldots s_{k_1} (s_{k_1} = u_2), P_{k_2+1} = u_1 t_1 t_2 \ldots t_{k_2} (t_{k_2} = u_2)$ connecting u_1, u_2, one internal path $P_{k_3+1} = u_1 x_1 x_2 \ldots x_{k_3} (x_{k_3} = v_1)$ connecting u_1, v_1, one internal path $P_{k_4+1} = u_2 y_1 y_2 \ldots y_{k_4} (y_{k_4} = v_2)$ connecting u_2, v_2, two internal paths $P_{k_5+1} = v_1 z_1 z_2 \ldots z_{k_5} (z_{k_5} = v_2), P_{k_6+1} = v_1 q_1 q_2 \ldots q_{k_6} (q_{k_6} = v_2)$ connecting v_1, v_2. Without loss of generality, We assume $k_2 \geq 2$. Note that $N_{\widetilde{G}}(u_1) = \{s_1, t_1, x_1\}$. If $k_3 = 1$, then $x_1 = v_1$; if $k_3 \geq 2$, then $x_1 \neq v_1$. Set $m = |\{x : d_G(x) = 3, x \in \{s_1, t_1, x_1\}\}|$. According to the structure of $\widetilde{G}$, we have $m = 0, 1, 2$. We consider the following two possible cases.

Case 1. There exists at least one vertex in $\{u_1, u_2, v_1, v_2\}$, say u_1, such that it is an attached vertex. In this case, we have $a + b - 1 \geq 3$.

If $d_G(t_1) = 2$, applying (1.1) at t_1 yields $d_G(t_2) = 2a + b - 7$; if $d_G(t_1) = a + b - 1$, applying (1.1) at t_1 yields $d_G(t_2) = -ab - b^2 + 2b - 1$.

Applying (1.1) at u_1, we get

$$d_G(s_1) + d_G(t_1) + d_G(x_1) = 3a + b - 9 \tag{3.31}$$

Subcase 1.1. $m = 0$, or 1. In this case, we have $\{d_G(s_1), d_G(t_1), d_G(x_1)\} = \{r, t \cdot 2, (2 - t) \cdot (a + b - 1)\}$, where $t = 0, 1, 2$ and $r \in \{2, 3, a + b - 1\}$. It is easy to see that $m = 1$ if $r = 3$ and 0 otherwise.

In view of (3.31), we have

$$r + 2t + (a + b - 1)(2 - t) = 3a + b - 9, \qquad t = 0, 1, 2 \tag{3.32}$$

• $t = 0$. Then (3.32) gives $a - b = 7 + r$. In this subcase we may assume $d_G(t_1) = a + b - 1$. Hence, we have

$$a - b = 7 + r \quad \text{and} \quad d_G(t_2) = -ab - b^2 + 2b - 1. \tag{3.33}$$

Note that $d_G(t_2) \in \{2, 3, a + b - 1\}$, $r \in \{2, 3, a + b - 1\}$. It is routine to check that (3.33) has no integer solution satisfying $a + b - 1 \geq 3$, a contradiction.

• $t=1$. Then at least one of $\{s_1,t_1,x_1\}$ is of degree two. We first assume that at least one vertex in $\{s_1,t_1\}$ is of degree 2. For convenience, let $d_G(t_1)=2$. On the other hand, since $t=1$, (3.32) gives $2a=10+r$. Hence,

$$2a=10+r \quad \text{and} \quad d_G(t_2)=2a+b-7 \tag{3.34}$$

with $d_G(t_2)\in\{2,3,a+b-1\}$. It's routine to check that only $r=2$ holds, which implies that $a=6$. If $d_G(t_2)=2$, then (3.34) gives $b=-3, a+b-1=2<3$, a contradiction. If $d_G(t_2)=3$, then (3.34) gives $b=-2$. Hence, $(a,b)=(6,-2)$. If $d_G(t_2)=a+b-1$, then $b\in\{-1,-2\}$ since $a+b-1\geq 3$ and $b\leq -1$.

First we consider $(a,b)=(6,-1)$ with $d_G(t_2)=a+b-1=4$. Note that $d_G(s_1)\in\{2,4\}$. If $d_G(s_1)=2$, then $d_G(x_1)=4$. Applying (1.1) at s_1 yields $d_G(s_2)=4$. Next, we will show that $s_2=t_2=u_2$. On the contrary, we suppose $s_2\neq t_2$. Applying (1.1) at s_2, we have $d_G(s_3)=3$, which implies that $s_3=u_2$ with $d_G(u_2)=3$. Hence, $t_2\neq u_2$ and $t_3=u_2$. Then applying (1.1) at u_2 yields $d_G(y_1)=0$, a contradiction. Therefore, we get that $s_2=u_2$, similarly, $t_2=u_2$. Thus, $d_G(u_2)=4$. Applying (1.1) at u_2 yields $d_G(y_1)=2$, moreover, $d_G(y_2)=4, d_G(y_3)=3$. Hence, $y_3=v_2$. Note that $d_G(x_1)=4$. It's easy to see that $x_1\neq v_1$. In fact, if $x_1=v_1$, then applying (1.1) at v_1 yields $d_G(z_1)+d_G(q_1)=3$. This is impossible. Therefore, we get that $x_1\neq v_1$. We apply (1.1) at x_1 to get $d_G(x_2)=2$, moreover, $d_G(x_3)=3$. So we have $x_3=v_1$ and $d_G(v_1)=3$. Applying (1.1) at v_1, we have $d_G(z_1)+d_G(q_1)=6$. Without loss of generality, we assume $d_G(z_1)=2, d_G(q_1)=4$. We apply (1.1) at z_1 and q_1 respectively to get $d_G(z_2)=4, d_G(q_2)=2$. Moreover, $d_G(z_3)=d_G(q_3)=3$. Hence, $z_3=q_3=v_2$. Thus, it's easy to check that $G\cong G_{33}\in\mathscr{G}_{6,-1}$; See Fig. 5.

If $d_G(s_1)=4$, then $d_G(x_1)=2$. By a similar proof above, we may also obtain that $G\cong G_{33}\in\mathscr{G}_{6,-1}$.

Now we consider $(a,b)=(6,-2)$ with $d_G(t_2)=3=a+b-1$. If $t_2\neq u_2$, then applying (1.1) at t_2 yields $d_G(t_3)=4\notin\{2,3\}$, a contradiction. Hence, $t_2=u_2$. Note that $d_G(s_1)\in\{2,3\}$. For $d_G(s_1)=2$, we may get $s_2=u_2$. By (1.1), it is routine to check that $d_G(x_1)=\cdots=d_G(x_{k_3})=3, d_G(y_1)=\cdots=d_G(y_{k_4})=3(k_3\geq 1, k_4\geq 1$ and $k_3k_4\geq 1)$. Furthermore, $d_G(z_1)=2, z_2=v_2; d_G(q_1)=2, q_2=v_2$. Therefore, we obtain that $G\cong G_{34}\in\mathscr{G}_{6,-2}$; see Fig. 5.

For $d_G(s_1)=3$, we have $d_G(x_1)=2$ and $x_2=v_1$. By (1.1), it is routine to check that $d_G(s_1)=\cdots=d_G(s_{k_1})=3(k_1\geq 1)$. Similarly, $d_G(y_1)=2$ and $y_2=v_2$; $d_G(z_1)=2, z_2=v_2; d_G(q_1)=\cdots=d_G(q_{k_6})=3(k_6\geq 1)$. Note that G has pendant vertices, then $k_1k_6\neq 1$). Hence, we obtain that $G\cong G_{35}\in\mathscr{G}_{6,-2}$; see Fig. 5.

Now we assume $d_G(t_1), d_G(s_1)\neq 2$. Hence, $d_G(x_1)=2$. By a similar discussion as the former subcase, we have $r=2$. Together with $t=1$, we have at least two of $\{s_1,t_1,x_1\}$ of degree 2, a contradiction to the assumption.

• $t=2$. Then (3.32) gives $3a+b=13+r$. Without loss of generality, we assume that $d_G(t_1)=2$. Hence,

$$3a+b=13+r \quad \text{and} \quad d_G(t_2)=2a+b-7. \tag{3.35}$$

Note that $d_G(t_2)\in\{2,3,a+b-1\}, r\in\{2,3,a+b-1\}$. It's easy to check that $r\neq 2, d_G(t_2)\neq 2$. Furthermore, for $d_G(t_2)=3$ with $r\in\{3,a+b-1\}$ or $d_G(t_2)=a+b-1$ with $r=3$, we have $(a,b)=(6,-2)$; for $d_G(t_2)=a+b-1$ with $r=a+b-1$, we have $a=6, b\in\{-1,-2\}$.

First we consider $(a,b)=(6,-1)$. In this subcase, $r=a+b-1=4$ and $\{d_G(s_1), d_G(t_1), d_G(x_1)\}=\{2,2,4\}$. By a similar discussion as in the proof of the case $t=1$ and $(a,b)=(6,-1)$, we can also obtain the graph $G\cong G_{33}\in\mathscr{G}_{6,-1}$.

Now we consider $(a,b)=(6,-2)$. In this subcase, $r=a+b-1=3$ and $\{d_G(s_1), d_G(t_1), d_G(x_1)\}=\{2,2,3\}$. By a similar discussion as in the proof of the case $t=1$ and $(a,b)=(6,-2)$, we can also obtain the graph $G\cong G_{34}, G_{35}\in\mathscr{G}_{6,-2}$.

Subcase 1.2. $m=2$, i.e., $\{d_G(s_1), d_G(t_1), d_G(x_1)\}=\{2,3,3\}$ or $\{d_G(s_1), d_G(t_1), d_G(x_1)\}=\{a+b-1,3,3\}$.

First we consider $\{d_G(s_1), d_G(t_1), d_G(x_1)\}=\{2,3,3\}$. In this subcase, we let $d_G(s_1)=2, d_G(t_1)=d_G(x_1)=3$. Thus, $t_1=u_2, x_1=v_1$ with $d_G(u_2)=d_G(v_1)=3$.

In view of (3.31), we have $2+3+3=3a+b-9$, i.e., $3a+b=17$. Applying (1.1) at s_1 yields $d_G(s_2)=2a+b-7\in\{2,3,a+b-1\}$. It's routine to check that only $d_G(s_2)=a+b-1$ holds, which implies $(a,b)=(6,-1)$. Note that $s_2\neq u_2$. Applying (1.1) at s_2 yields $d_G(s_3)=3$, thus, $s_3=u_2$. Applying (1.1) at u_2, we get that $d_G(y_1)=1$, a contradiction.

Now consider that $\{d_G(s_1),d_G(t_1),d_G(x_1)\}=\{a+b-1,3,3\}$. In this subcase, we let $d_G(s_1)=a+b-1, d_(t_1)=d_G(x_1)=3$. Thus, $t_1=u_2, x_1=v_1$ with $d_G(u_2)=d_G(v_1)=3$.

In view of (3.31), we have $a+b-1+3+3=3a+b-9$, which gives that $a=7$. Applying (1.1) at s_1 yields $d_G(s_2)=-ab-b^2+2b-1\in\{2,3,a+b-1\}$. It's routine to check that only $d_G(s_2)=3$ holds, which implies $(a,b)=(7,-1)$ and $s_2=u_2$. Applying (1.1) at u_2, we get $v_2\in N_G(u_2)$ and $d_G(v_2)=3$. Applying (1.1) at v_1 yields $d_G(z_1)+d_G(q_1)=8$. Without loss of generality, we assume $d_G(z_1)=3, d_G(q_1)=5$. Thus, $z_1=v_2$. Applying (1.1) at q_1, we have $d_G(q_2)=3$. Hence, $q_2=v_2$. It's simple to check that $G\cong G_{36}\in\mathscr{G}_{7,-1}$; See Fig. 5.

Case 2. Each of the vertices in $\{u_1,\, u_2,\, v_1,\, v_2\}$ is an attached-vertex. That is, $d_G(u_1)=d_G(u_2)=d_G(v_1)=d_G(v_2)=a+b-1>3$. In this case, we have $\{d_G(s_1),d_G(t_1),d_G(x_1)\}=\{t\cdot 2,(3-t)\cdot(a+b-1)\}$, where $0\le t\le 3$.

If $d_G(t_1)=2$, applying (1.1) at t_1, we have $d_G(t_2)=a-3\in\{2,a+b-1\}$. Since $a\geq 6$, we obtain $a-3\neq 2$. For $d_G(t_2)=a+b-1$, we have $b=-2$. If $d_G(t_1)=a+b-1$, applying (1.1) at t_1, we have $d_G(t_2)=-ab-b^2-a+b+3$.

Applying Lemma 1.8(ii) at u_1, we get $d_G(s_1)+d_G(t_1)+d_G(x_1)=-ab-b^2+2b+3$, which gives

$$2t+(a+b-1)(3-t)=-ab-b^2+2b+3,\qquad t=0,1,2,3. \tag{3.36}$$

• $t=0$. (3.36) gives $ab+b^2+3a+b=6$. In this subcase, we have $d_G(t_1)=a+b-1$. Hence,

$$ab+b^2+3a+b=6\quad\text{and}\quad d_G(t_2)=-ab-b^2-a+b+3$$

with $d_G(t_2)\in\{2,a+b-1\}$. It's routine to check that there is no integer solution such that $a+b-1>3$.

• $t=1$. (3.36) gives $ab+b^2+2a=3$. If $d_G(t_1)=2$, then $b=-2$. However, it does not satisfy the equation. Therefore, $d_G(t_1)=a+b-1$. Hence,

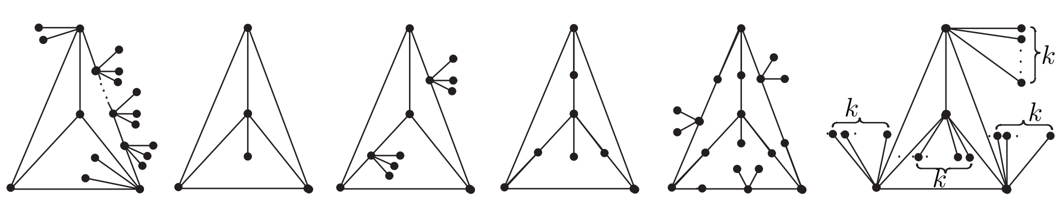
$$ab+b^2+2a=3\quad\text{and}\quad d_G(t_2)=-ab-b^2-a+b+3. \tag{3.37}$$

By (3.37), we get that $d_G(t_2)=a+b\in\{2,a+b-1\}$. It's impossible since $a+b-1>3$.

• $t=2,3$. Without loss of generality we assume $d_G(t_1)=2$. Then we have $b=-2$. For $t=2$, (3.36) gives $ab+b^2+a-b=0$; for $t=3$, (3.36) gives $ab+b^2-2b+3=0$. It's easy to check that both are impossible since $b=-2$.

Thus, we complete the proof. $\square$

Proposition 7. *Let $G\in\mathscr{G}_{a,b}$ be a tricyclic graph with $\widetilde{G}=T_{15}$. Then $G\cong G_{34}\in\mathscr{G}_{8,-2}$, or $G\cong G_{35}\in\mathscr{G}_{7,-2}$, or $G_{36}\in\mathscr{G}_{7,-1}$, or $G\cong G_{37}\in\mathscr{G}_{a,b}$ with $\frac{6-3a}{b}=k\geq 6$, or $G\cong G_{38}\in\mathscr{G}_{8,-2}$, or $G\cong G_{39}\in\mathscr{G}_{6,-1}$, or $G\cong G_{40}\in\mathscr{G}_{6,-1}$, where $G_{34},G_{35},G_{36},G_{37}$ and G_{38},G_{39},G_{40} are depicted in Fig. 5.*



$G_{37}\in\mathscr{G}_{8,-2}$ $G_{38}\in\mathscr{G}_{7,-2}$ $G_{39}\in\mathscr{G}_{7,-1}$ $G_{40}\in\mathscr{G}_{6,-1}$ $G_{41}\in\mathscr{G}_{6,-1}$ $G_{42}\in\mathscr{G}_{a,b}$

Figure 6: Graphs $G_{37},G_{38},G_{39},G_{40},G_{41}$ and G_{42}.

Proof. Note that $\widetilde{G} = T_{15}$ (see Fig. 1), then $d_{\widetilde{G}}(v_1) = d_{\widetilde{G}}(v_2) = d_{\widetilde{G}}(v_3) = d_{\widetilde{G}}(v_4) = 3$ and $\widetilde{G}$ consists of six internal paths: $P_{k_1} = v_2u_1 \dots u_{k_1}(u_{k_1} = v_3), P_{k_2} = v_2w_1 \dots w_{k_2}(w_{k_2} = v_4), P_{k_3} = v_3s_1 \dots s_{k_3}(s_{k_3} = v_4), P_{k_4} = v_1x_1 \dots x_{k_4}(x_{k_4} = v_2), P_{k_5} = v_1y_1 \dots y_{k_5}(y_{k_5} = v_3), P_{k_6} = v_1z_1 \dots z_{k_6}(z_{k_6} = v_4)$.

Case 1. There exists at least one vertex in $\{v_1, v_2, v_3, v_4\}$, say v_1, such that it is a non-attached vertex. That is, $d_G(v_1) = 3$. In this case, $a + b - 1 \geq 3$. Let $N_{\widetilde{G}}(v_1) = \{r_1, r_2, r_3\}$, then $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{t_1 \cdot 2, t_2 \cdot 3, t_3 \cdot (a + b - 1)\}$, where $t_1 + t_2 + t_3 = 3$ with $0 \leq t_i \leq 3 (i = 1, 2, 3)$.

Subcase 1.1. $t_1 = 0$, that is $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{t \cdot 3, (3 - t) \cdot (a + b - 1)\}$, where $0 \leq t \leq 3$.

Applying (1.1) at v_1 yields $d_G(r_1) + d_G(r_2) + d_G(r_3) = 3a + b - 9$. Hence we have

$$3t + (a + b - 1)(3 - t) = 3a + b - 9, \qquad (t = 0, 1, 2, 3). \tag{3.38}$$

• $t = 0$. Then (3.38) gives that $b = -3$. First we assume that $v_2, v_3, v_4 \in N_G(v_1)$, then $d_G(v_2) = d_G(v_3) = d_G(v_4) = a + b - 1 > 3$. Applying (1.1) at v_2 yields that $d_G(u_1) + d_G(w_1) = -ab - b^2 + 2b$, whence we may check that $d_G(u_1), d_G(w_1)$ will not be in $\{2, a + b - 1\}$, a contradiction.

So we assume that at least one member in $\{v_2, v_3, v_4\}$, say v_4, is not in $N_G(v_1)$. Thus, $d_G(z_1) = a + b - 1 > 2$. Applying Lemma (1.8)(ii) at z_1 yields $d_G(z_2) = -ab - b^2 + 2b - 1 \in \{2, 3, a + b - 1\}$. It's routine to check that there is no integer solution such that $a + b - 1 > 2$, a contradiction.

• $t = 1$. Without loss of generality, we suppose $d_G(v_2) = 3$ with $v_2 \in N_G(v_1)$. By (3.38), we have $a - b = 10$.

If $v_3, v_4 \in N_G(v_1)$, thus, $d_G(v_3) = d_G(v_4) = a + b - 1 > 3$. Applying (1.1) at v_2 yields that $d_G(u_1) + d_G(w_1) = 3a + b - 12$. Notice that $d_G(u_1), d_G(w_1) \in \{2, a + b - 1\}$, it is easy to check that only $d_G(u_1) = d_G(w_1) = a + b - 1$ is true, which also implies that $a - b = 10$.

We first assume $u_1 = v_3$ and $w_1 = v_4$. Then applying (1.1) at v_3 yields $d_G(s_1) = -ab - b^2 + 2b - 3 \in \{2, a + b - 1\}$. Note that $a - b = 10$, for $d_G(s_1) = 2$, there is no integer solution; for $d_G(s_1) = a + b - 1$, we get that $(a, b) = (8, -2)$ and $a + b - 1 = 5$. It's easy to check that $d_G(s_i) = 5$ for $i = 1, 2, \dots, k_3$, where $k_3 \geq 1$. Thus, we get $G \cong G_{37} \in \mathscr{G}_{8,-2}$; see Fig. 6.

Now we assume, without loss of generality, that $w_1 \neq v_4$. Then applying (1.1) at w_1 yields $d_G(w_2) = -ab - b^2 + 2b - 1 \in \{2, a + b - 1\}$. It is easy to check that this is impossible.

If $\{v_3, v_4\}$ contains a member, say v_4, not in $N_G(v_1)$. Thus, $d_G(z_1) = a + b - 1$ with $z_1 \neq v_4$. Applying (1.1) at z_1 yields $d_G(z_2) = -ab - b^2 + 2b - 1 \in \{2, 3, a + b - 1\}$. It is routine to check that this is impossible.

• $t = 2$. Without loss of generality, we assume $d_G(v_2) = d_G(v_3) = 3$ with $v_2, v_3 \in N_G(v_1)$. By (3.38) we get $a = 7$. First, we consider $v_4 \in N_G(v_1)$, then $d_G(v_4) = a + b - 1 > 3$. Applying (1.1) at v_2 gives $d_G(u_1) + d_G(w_1) = 3a + b - 12$. Note that $d_G(u_1) \in \{2, 3, a + b - 1\}, d_G(w_1) \in \{2, a + b - 1\}$. We may check that only $d_G(u_1) = 3$ with $u_1 = v_3$ and $d_G(w_1) = a + b - 1$ is true, which also implies that $a = 7$.

If $w_1 \neq v_4$, applying (1.1) at w_1, we have that $d_G(w_2) = -ab - b^2 + 2b - 1 \notin \{2, a + b - 1\}$, a contradiction. So we have $w_1 = v_4$. Similarly, by applying (1.1) at v_3, we may get $d_G(s_1) = a + b - 1$ and $s_1 = v_4$. Thus, we get the graph $G \cong G_{39}$. We apply Lemma 1.8(ii) at v_4 to get $3 + 3 + 3 = -ab - b^2 + 2b + 3$, together with $a = 7$, we have $b = -2$. Hence $G \cong G_{38} \in \mathscr{G}_{7,-2}$; see Fig. 6.

Now we consider $v_4 \notin N_G(v_1)$, thus, $d_G(z_1) = a + b - 1$ with $z_1 \neq v_4$. Applying (1.1) at z_1 yields $d_G(z_2) = -ab - b^2 + 2b - 1 \in \{2, 3, a + b - 1\}$. It is routine to check that only $d_G(z_2) = 3$ holds, which implies $b = -1$ since $a + b - 1 > 3$. Therefore, $z_2 = v_4$ and $d_G(v_4) = 3$. Note that $(a, b) = (7, -1), a + b - 1 = 5$. Applying (1.1) at v_2, we get that $d_G(u_1) + d_G(w_1) = 8$. If $d_G(w_1) = 5, d_G(u_1) = 3$ with $u_1 = v_3$, then $d_G(w_2) = 3$. Hence, $w_2 = v_4$. Then applying (1.1) at v_4 yields $d_G(s_{k_3-1}) = 1$, it's impossible. So we have $d_G(u_1) = 5, d_G(w_1) = 3$ with $w_1 = v_4$. Moreover, $d_G(u_2) = 3$, so $u_2 = v_3$. Applying (1.1) at v_3 yields $v_4 \in N_G(v_3)$. Thus, we obtain the graph $G \cong G_{39} \in \mathscr{G}_{7,-1}$; see Fig. 6.

• $t = 3$. Thus, $v_2, v_3, v_4 \in N_G(v_1)$ and $d_G(v_2) = d_G(v_3) = d_G(v_4) = 3$. By (3.38) we have $3a + b = 18$. Then applying (1.1) at v_2 yields $d_G(u_1) + d_G(w_1) = 3a + b - 12 = 6$. Note that $d_G(u_1), d_G(w_1) \in \{2, 3, a + b - 1\}$, it is routine to check that $d_G(u_1), d_G(w_1) \neq 2$. Together with $d_G(u_1) + d_G(w_1) = 6$, we have $d_G(u_1) = d_G(w_1) = 3$.

First we assume $u_1 = v_3, w_1 = v_4$. Applying (1.1) at v_3, we get $d_G(s_1) = 3$. If $s_1 = v_4$, thus G is a regular graph with only one Q-main eigenvalue, a contradiction. So we have $s_1 \neq v_4$. Hence, $a + b - 1 = d_G(s_1) = 3$. It implies that $(a, b) = (7, -3)$. Applying (1.1) at s_1, we have $d_G(s_2) = 5 \notin \{2, 3\}$, a contradiction.

Now we assume, without loss of generality, that $w_1 \neq v_4$. We may also get a contradiction by a similar discussion as above.

Subcase 1.2. $t_1 = 1$, that is $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{2, t \cdot 3, (2 - t) \cdot (a + b - 1)\}$, where $t = 0, 1, 2$.

In this case, we assume, without loss of generality, that $d_G(x_1) = 2$.

Applying (1.1) at v_1 yields

$$2 + 3t + (a + b - 1)(2 - t) = 3a + b - 9. \tag{3.39}$$

Applying (1.1) at x_1, we get that $d_G(x_2) = 2a + b - 7 \in \{2, 3, a + b - 1\}$.

It's routine to check that only $t = 2$ with $d_G(x_2) = a + b - 1$ holds. It implies that $d_G(v_3) = d_G(v_4) = 3$ and $(a, b) = (6, -1)$. Hence, $d_G(x_2) = 4$.

First we consider $x_2 \neq v_2$. Applying (1.1) at x_2 yields $d_G(x_3) = 3$. Hence, we have $x_3 = v_2$ with $d_G(v_2) = 3$. We apply (1.1) at v_2 to get $d_G(u_1) + d_G(w_1) = 4$. It implies that $d_G(u_1) = d_G(w_1) = 2$. Applying (1.1) at u_1, we have $d_G(u_2) = 4$, moreover, $d_G(u_3) = 3$ with $u_3 = v_3$. Then applying (1.1) at v_3 yields $d_G(s_1) = 1$, it's impossible.

Now we consider $x_2 = v_2$. Thus, $d_G(v_2) = 4$. Applying Lemma 1.8(ii) at v_2 yields $d_G(u_1) + d_G(w_1) = 4$, which implies that $d_G(u_1) = d_G(w_1) = 2$. Applying (1.1) at u_1 and w_1 respectively, we have $d_G(u_2) = d_G(w_2) = 3$ with $u_2 = v_3$ and $w_2 = v_4$. Then applying (1.1) at v_3 gives $v_4 \in N_G(v_3)$. Thus, it's simple to check that $G \cong G_{40} \in \mathscr{G}_{6,-1}$; see Fig. 6.

Subcase 1.3. $t_1 = 2$, that is $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{2, 2, 3\}$, or $\{2, 2, a + b - 1\}$.

In this case, we assume, without loss of generality, that $d_G(x_1) = d_G(y_1) = 2$.

First consider $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{2, 2, 3\}$. Then applying (1.1) at v_1 yields $2 + 2 + 3 = 3a + b - 9$. It implies that $3a + b = 16$. Applying (1.1) at x_1 yields $d_G(x_2) = 2a + b - 7 \in \{2, 3, a + b - 1\}$. It's easy to check that $d_G(x_2) \neq 2$. For $d_G(x_2) \in \{3, a + b - 1\}$, we have $(a, b) = (6, -2)$ and $a + b - 1 = 3$. If $x_2 \neq v_2$, then applying (1.1) at x_2 yields $d_G(x_3) = 4 \notin \{2, 3\}$, a contradiction. Hence, $x_2 = v_2$. By a similar proof, we finally obtain $G \cong G_{24} \in \mathscr{G}_{6,-2}$, which has no pendant vertices. It's a contradiction.

Now we consider $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{2, 2, a + b - 1\}$. Then applying (1.1) at v_1 yields $2 + 2 + a + b - 1 = 3a + b - 9$, which gives $a = 6$. Since $a + b - 1 \geq 3$, we have $b = -1$, or -2.

If $(a, b) = (6, -2)$, then $a + b - 1 = 3$. Thus, $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{2, 2, 3\}$. By a similar discussion as above, this is impossible.

If $(a, b) = (6, -1)$, then $d_G(z_1) = a + b - 1 = 4$. If $z_1 = v_4$, then applying (1.1) at v_4 yields $d_G(s_{k_3-1}) + d_G(w_{k_2-1}) = 3$, it's impossible. Hence, $z_1 \neq v_4$. By (1.1) we can get that $d_G(z_2) = 2, d_G(z_3) = 3$. Thus, $z_3 = v_4$ with $d_G(v_4) = 3$. Applying (1.1) at v_4 yields $d_G(s_{k_3-1}) + d_G(w_{k_2-1}) = 6$. Hence, either $d_G(s_{k_3-1}) = d_G(w_{k_2-1}) = 3$, or $\{d_G(s_{k_3-1}), d_G(w_{k_2-1})\} = \{2, 4\}$.

If $d_G(s_{k_3-1}) = d_G(w_{k_2-1}) = 3$, then $v_2, v_3 \in N_G(v_4)$ and $d_G(v_2) = d_G(v_3) = 3$. Applying (1.1) at x_1 yields $d_G(x_2) = 4$, moreover, $d_G(x_3) = 3$. Hence, $x_3 = v_2$. Similarly, $d_G(y_2) = 4, d_G(y_3) = 3$, and $y_3 = v_3$. Applying (1.1) at v_2 yields $d_G(u_1) = 1$, a contradiction.

Hence, we consider $\{d_G(s_{k_3-1}), d_G(w_{k_2-1})\} = \{2, 4\}$. Without loss of generality, we assume $d_G(s_{k_3-1}) = 2, d_G(w_{k_2-1}) = 4$. Then applying (1.1) at w_{k_2-1} yields $d_G(w_{k_2-2}) = 2$, moreover, $d_G(w_{k_2-3}) = 3$. Thus, we have $w_{k_2-3} = v_2$ with $d_G(v_2) = 3$. We apply (1.1) at x_1 to get $d_G(x_2) = 4$, moreover, $d_G(x_3) = 3$ with $x_3 = v_2$. Then

applying (1.1) at v_2 yields $d_G(u_1) = 2$. Note that $d_G(u_1) = d_G(y_1) = d_G(s_{k_3-1}) = 2$, then applying (1.1) at u_1, y_1, s_{k_3-1} respectively yields $d_G(u_2) = d_G(y_2) = d_G(s_{k_3-2}) = 4$.

If $u_2 \neq v_3$, then applying (1.1) at u_2 yields $d_G(u_3) = 3$, which implies that $u_3 = v_3$ and $d_G(v_3) = 3$. Thus, we have $y_2 \neq v_3$, $s_{k_3-2} \neq v_3$. By a similar discussion as u_2, we may get that $y_3 = s_{k_3-3} = v_3$. However, by applying (1.1) at v_3, we have $4+4+4 = 3a+b-9$ with $(a,b) = (6,-1)$. It's a contradiction. Therefore, $u_2 = v_3$. Similarly, $y_2 = s_{k_3-2} = v_3$. Thus, it's easy to check that $G \cong G_{41} \in \mathscr{G}_{6,-1}$; see Fig. 6.

Subcase 1.4. $t_1 = 3$, that is $d_G(r_1) = d_G(r_2) = d_G(r_3) = 2$.

In this case, $d_G(x_1) = d_G(y_1) = d_G(z_1) = 2$.

Applying (1.1) at v_1, we have $2+2+2 = 3a+b-9$, i.e., $3a+b = 15$. Applying (1.1) at x_1 yields $d_G(x_2) = 2a+b-7 \in \{2, 3, a+b-1\}$. It's easy to check that this is impossible.

Case 2. Each of the vertices in $\{v_1, v_2, v_3, v_4\}$ is an attached-vertex. That is, $d_G(v_1) = d_G(v_2) = d_G(v_3) = d_G(v_4) = a+b-1 > 3$. Let $N_{\widetilde{G}}(v_1) = \{r_1, r_2, r_3\}$, then $\{d_G(r_1), d_G(r_2), d_G(r_3)\} = \{t \cdot 2, (3-t) \cdot (a+b-1)\}$, where $0 \leq t \leq 3$.

Applying Lemma 1.8(ii) at v_1, we have $d_G(r_1) + d_G(r_2) + d_G(r_3) = -ab - b^2 + 2b + 3$. Hence,

$$3t + (a+b-1)(3-t) = -ab - b^2 + 2b + 3. \tag{3.40}$$

• $t = 0$. By (3.40) we have $ab + b^2 + 3a + b = 6$. If $v_2, v_3, v_4 \in N_G(v_1)$, we apply (1.1) at v_2 to get $d_G(u_1) + d_G(w_1) = -ab - b^2 - a + b + 4$. Note that $d_G(u_1), d_G(w_1) \in \{2, a+b-1\}$. It is routine to check that only $d_G(u_1) = d_G(w_1) = a+b-1$ is true.

First we assume that $u_1 = v_3, w_1 = v_4$. Applying (1.1) at v_3, we have $d_G(s_1) = a+b-1$. By (1.1), it's easy to check that $s_1 = v_4$. Hence, we obtain the graph $G \cong G_{42} \in \mathscr{G}(a,b)$, where a, b satisfying $ab + b^2 + 3a + b = 6$, with $a+b-1 \geq 4$ and $b \leq -1$. Denote the number of pendant vertices attached at $v_i (i = 1,2,3,4)$ by k, then $k = a+b-1-3 \geq 1$; see Fig. 6.

Now we assume, without loss of generality, $w_1 \neq v_4$. Applying (1.1) at w_1, we get that $d_G(w_2) = -ab - b^2 - a + b + 3 \in \{2, a+b-1\}$, which gives no integer solution such that $a+b-1 > 3$, contradiction.

If $\{v_2, v_3, v_4\}$ contains a member, say v_4, not in $N_G(v_1)$. Thus, $d_G(z_1) = a+b-1$ with $z_1 \neq v_4$. Applying (1.1) at z_1 yields $d_G(z_2) = -ab - b^2 - a + b + 3 \in \{2, a+b-1\}$. It is routine to check that this is impossible.

• $t = 1, 2, 3$. In this subcase, we assume, without loss of generality, that $d_G(x_1) = 2$. Applying (1.1) at x_1, we get that $d_G(x_2) = a - 3 \in \{2, a+b-1\}$. It's easy to check that $d_G(x_2) \neq 2$. So we consider $d_G(x_2) = a+b-1$. It implies that $b = -2$. For $t = 1$, (3.40) gives $ab + b^2 + 2a = 3$; for $t = 2$, (3.40) gives $ab + b^2 + a - b = 0$; for $t = 3$, (3.40) gives $ab + b^2 - 2b + 3 = 0$. Each gives no integer solution such that $a+b-1 > 3$ since $b = -2$. It's a contradiction.

Thus, we complete the proof. □

Theorem 3.3. *$G_{28} \in \mathscr{G}_{6,-1}, G_{29} \in \mathscr{G}_{6,-1}, G_{30} \in \mathscr{G}_{6,-1}, G_{31} \in \mathscr{G}_{6,-1}, G_{32} \in \mathscr{G}_{7,-2}, G_{33} \in \mathscr{G}_{6,-1}, G_{34} \in \mathscr{G}_{6,-2}, G_{35} \in \mathscr{G}_{6,-2}, G_{36} \in \mathscr{G}_{7,-1}, G_{37} \in \mathscr{G}_{8,-2}, G_{38} \in \mathscr{G}_{7,-2}, G_{39} \in \mathscr{G}_{7,-1}, G_{40} \in \mathscr{G}_{6,-1}, G_{41} \in \mathscr{G}_{6,-1}, G_{42} \in \mathscr{G}_{a,b}$ (see Figs. 3-6) are all the tricyclic graphs with pendants having exactly two Q-main eigenvalues.*

Proof. By Propositions 2,3 and 5, $G \notin \mathscr{G}_{a,b}$ if $\widetilde{G} = T_3, T_9, T_{13}$. In view of the proof of Proposition 1, we obtain that $G \notin \mathscr{G}_{a,b}$ if $\widetilde{G} = T_1, T_2, T_5, T_7, T_8, T_{10}$. Hence, our results follow immediately from Propositions 1, 4, 6, and 7. □

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